Calculating a Derivative via the Limit Definition

We wish to calculate the derivative using the limit definition of the function $f(x) = x^{-2}$.

We first note the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

So, in the case of $f(x) = x^{-2}$, we have

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{(x+h)^{-2} - (x)^{-2}}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \left(\frac{1}{(x+h)^2} - \frac{1}{x^2} \right) \\
&= \lim_{h \to 0} \frac{1}{h} \left(\frac{x^2 - (x+h)^2}{(x+h)^2 \cdot x^2} \right) \\
&= \lim_{h \to 0} \frac{1}{h} \left(\frac{x^2 - (x^2 + 2hx + h^2)}{(x+h)^2 \cdot x^2} \right) \\
&= \lim_{h \to 0} \frac{1}{h} \left(\frac{-2hx - h^2}{(x+h)^2 \cdot x^2} \right) \\
&= \lim_{h \to 0} \frac{-2x - h}{(x+h)^2 \cdot x^2} \\
&= \frac{-2x}{x^4} \\
&= \frac{-2}{x^3}
\end{aligned}$$

In the third line, we simply put the fractions of the above line over a common denominator. The rest is (hopefully) fairly self-explanatory.