The Product Rule

A common mistake is not to use the Product Rule. Quite often, the mistake is generally to try to say that $(fg)' = f'g'$, however this is NOT true! The Product Rule is as follows:

$$(fg)' = f'g + g'f$$

It turns out, that if $(fg)' = f'g'$ were true, the world would be very boring. In fact, if this were true every derivative (ever) would be zero. Notice that for any function $f(x) = 1 \cdot f(x)$, so trying to use the BAD product rule, we would have $f'(x) = (1)'f'(x) = 0$.

Proof (Product Rule)
We can establish the product rule directly from the limit definition of the derivative.

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) + f(x+h)g(x) - f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} f(x+h)\left(\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right) + g(x)\left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right) \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

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