Infinite Series for Math 230

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What is a Series?

A series can be defined as the sum of the terms of a sequence. An infinite series is the sum of the terms of an infinite sequence. It usually looks something like this: $\sum_{n=0}^{\infty} a_n$. We must be careful to distinguish between series and sequences. We can have an infinite sequence $\{a_n\}$ that converges, with the infinite series $\sum_{n=0}^{\infty} a_n$ diverging.

We define the n^{th} partial sum, s_n by $\sum_{j=0}^{n} a_n$.

We say that an infinite series converges if the sequence of partial sums of the series converge. Further, if the partial sums do converge, the series converges to the same number.

Side Note: It is very important to keep the ideas of sequences and series straight, they are two different mathematical objects that are closely related. Remember that a series is the summation of a sequence, and a sequence is a group of numbers put "in order" somehow.

n^{th} Term Test

If $\lim_{n\to\infty} a_n \neq 0$, or this limit does not exist, (for instance, in the case of an oscillating or diverging sequence) then the infinite series $\sum a_n$ diverges.

Geometric Series

The geometric series is one of our best tools for determining not just if a sequence converges, but in finding the real number to which our series converges. A geometric series is a series where each term can be defined be: $a_{n+1} = ra_n$ for some given, constant r. A geometric series converges if $|r| < 1$, if $|r| \geq 1$ our sequence diverges.

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

The Harmonic Series

The other "famous" series we will get to know very well is the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \infty$. That is, the Harmonic Series diverges. Also, any multiple of a harmonic series or any series with a "harmonic part" will also diverge. Here, we have an example of when the n^{th} Term Test goes to zero, but the series still diverges. This shows us that the n^{th} Term Test is a necessary condition for a series to converge, but by itself does not tell us whether a series will converge.

How to Use these Series

If we a series that is a linear combination of two sequences whose infinite series converge, our new series also converges. In particular, if: $\sum a_n = A$, and $\sum b_n = B$ and c, d are real numbers, then

$$\sum ca_n \pm db_n = cA \pm dB$$

Note that things can get pretty ugly (and usually wrong!) if you try to use these rules breaking up a series into different divergent series.