Power Series for Math 230

William Green

What is a Power Series?

A power series is an "infinite polynomial," specifically one that can be represented nicely in a series. Technically speaking, a power series is an infinite series of the form:

$$\sum_{n=0}^{\infty} a_n x^n$$

We have actually already encountered a few power series, for instance:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+!)}$$

When does a Power Series Converge?

If we are give a power series we have three distinct cases for when the series can converge. It can converge for all values of x, it can converge only when $x = 0$, which gives us the zero series and a zero polynomial, so this case isn't all that interesting, or we can have some real number R that is called the "Radius of Convergence" where x converges on some interval $|x - x_0| < R$, called the "Interval of Convergence."

Determining when a power series converges is closely related to the Ratio Test. Remembering how we define ρ, we remember that the series converges absolutely when $\rho < 1$, so, this carries over quite naturally to determining the radius of convergence of the power series. So, given the power series $\sum_{n=0}^{\infty} a_n x^n$, it converges when: $\lim_{n\to\infty} |\frac{a_{n+1}x^{n+1}}{a_n x^n}| < 1$.

An alternative statement is that the series converges when $|x| < R$, where R is defined by: $R = \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}|$. But, we recall from the ratio test that a series can still converge if $\rho = 1$, so we have to test the endpoints of our radius of convergence. To do this, we simply plug in the values of the ends of our interval of convergence for x and see what happens to the series.

Power Series Not based at $x = 0$

We spoke about the interval of convergence being about some basepoint x_0, we do this to account for the cases when we have a power series of the form:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Here x_0 is simply some fixed real number. Series of this type will have radii of convergence that are not centered about $x = 0$ as some of our nicer examples do.

Power Representations of Functions

There are often encountered functions that are just easier to deal with in terms of their power series. When a function $f(x)$ can be written as: $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we say that this is a power series representation of $f(x)$. These should seem fairly familiar as we have in fact done several examples of these when using Taylor Series of functions. A Taylor Series is just a special case of a power series where $a_n = \frac{f^{(n)}(a)}{n!}$.

The Binomial Series

The binomial series is a very famous, and very useful series. We remember the Binomial Theorem, which states that:

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

The binomial series is the case of the binomial theorem where $y = 1$ and n is any real number. (This is only interesting if n is not an integer.) So,

$$(1 + x)^n = 1 + \sum_{k=1}^{\infty} \frac{n(n-1)(n-2)\cdots(n-k+1)}{n!} x^n$$

A few calculations show that this binomial series has a radius of convergence about zero of length one. That is $|x| < 1$ or $(-1, 1)$ is the interval of convergence of this series.

Differentiating and Integrating Power Series

A very nice, and helpful theorem tells us that if we have a power series representation of a function, we can differentiate the function by differentiating term by term in the power series. Similarly, we can integrate the function by integrate term by term in the power series. In math this is: Given a function and an power series representation, $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we can determine the derivative to be $f^{'}(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and the integral to be $\int_{x_0}^{x} f(t)dt = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ as long as we are in the interval of convergence of our original function.