

Power Series Computations for Math 230

William Green

Binomial Series

Alternating Power Series, for instance the sine and cosine series, can be estimated to a given precision by use of the partial sums. In particular, the remainder of an alternating, decreasing series for the n^{th} partial sum is always less than or equal to (in absolute value) the $n + 1^{\text{th}}$ term. That is, we can say with $E = \sum_{k=0}^{\infty} (-1)^{k+1} a_k - \sum_{k=0}^n (-1)^{k+1} a_k$, $|E| < a_{n+1}$.

We will often use this idea with the binomial series. We know that:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + -\frac{5}{128}x^4 + \dots$$

The idea is, that now whenever we are given $\sqrt[n]{y}$ where y is not a power of n , we can calculate a fairly reasonable estimate of the value. First, we need to break up y into the sum of a power of n and a smaller number. So, $y = z^n + d$, we can rewrite $\sqrt[n]{y}$ as $\sqrt[n]{z^n + d} = \sqrt[n]{z^n(1 + \frac{d}{z^n})} = z \sqrt[n]{1 + \frac{d}{z^n}}$. Now we can plug this into the first few terms of our binomial series and get a fairly reasonable estimate.

$$\sqrt[n]{y} = z \sqrt[n]{1 + \left(\frac{d}{z^n}\right)} = z\left(1 + n\frac{d}{z^n} - n(n-1)\left(\frac{d}{z^n}\right)^2 + n(n-1)(n-2)\left(\frac{d}{z^n}\right)^3 + \dots\right)$$

Similarly, we can numerically estimate integrals that we can not integrate directly. For instance, e^{-x^2} cannot be integrated, but we can get approximate answers when integrating terms of the power series in our interval of convergence.

Adding and Multiplying Power Series

Adding and subtracting power series is just as simple and straightforward as we could hope for.

$$\sum_{n=0}^{\infty} a_n \pm \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} a_n \pm b_n$$

However, we need to be more careful when multiplying power series. We know that if we multiply two polynomials together, that the coefficient of any power of x depends on more than just the coefficients of x in our two polynomials. For instance, the coefficient on x^2 in the product of $x^2 + 2x + 3$ and $x^4 + x^2 + 2x - 4$ is $1 \cdot (-4) + 2 \cdot 2 + 3 \cdot 1 = 3$. So, our guess would be that the coefficient on x^n depends on all the terms in both power series of the first two functions whose power of x is less than or equal to n .

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

Where $c_n = \sum_{k=0}^n a_k b_{n-k}$

Power Series as an Alternative to L'Hôpital

Another use of Power Series is to replace functions in an indeterminate form with their power series and have a fraction defined wholly in terms of powers of x . This is often a lot easier, as we often get a lot of cancellation and other good things to happen.