Taylor Polynomials and Series for Math 230

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Using Polynomials to Approximate a Function

There often arise situations in mathematics, physics, engineering or just life in general where we want to be able to approximate the value of a function with a simpler function, such as a polynomial. The simplest way we can do this is akin to Newton's Method. We can say that for values of x "close enough" to a chosen basepoint a, that $f(x) \approx f(a) + f^{'}(a)(x-a)$. We can see that this will provide us with a good approximation only when x is very close to a.

Perhaps we want to be able to better approximate our function further away, the next term we would want to add would be something to make our approximation function a quadratic, and we probably want to it to be somehow weighted by the 2^{nd} derivative of our function at a.

Taylor Polynomials

This is where the idea a Taylor Polynomial comes in. Suppose that the first n derivatives of our function $f(x)$ exist at $x = a$, we define the n^{th} Order Taylor Polynomial by:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

And now this polynomial is a good approximation of $f(x)$ around $x = a$.

A Taylor Polynomial can be made to be exactly equal to our function with a slight modification, that is, $f(x) = P_n(x) + R_n(x)$ when we define our Taylor Remainder Function by:

$$R_n(x) = \frac{f^{n+1}(z)}{(n+1)!}(x-a)^{n+1}$$

Where we have some number z between a and x with a our basepoint for the Taylor Polynomials. (We know nothing about finding this value of z, similar to the big Calculus I result the Mean Value Theorem.)

Taylor Series

The main problem with using these Taylor Polynomials in practice is that we often want an exact solution, but we often have no way of determining what value to chose for z in our Remainder Function that will make our Taylor Polynomial an exact answer. When we have functions that are "nice enough" to let $\lim_{n\to\infty} R_n(x) = 0$, and we often do, we can use a Taylor Series to exactly identify our function as a "polynomial" of infinite degree.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor Series are most often based about $x = 0$ and these are often called Maclaurin series, which are just a case of Taylor Series. There are three Taylor Series that you should know offhand:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

We can remember the fact that cosine is an "even" function and then notice that the Taylor Series for cosine is composed of only even powers of x. And sine is an "odd" function and its Taylor Series is composed of only odd powers of x.

Euler's Formula

The book decides to throw in Euler's Formula at this point. Euler's Formula states that $e^{i\theta} = \cos\theta + i\sin\theta$, remembering that $i = \sqrt{-1}$. We note that this is a handy formula for deriving a lot of trigonometric identities. We use DeMoivre's Formula, which is just a special case of Euler's Formula, $e^{in\theta} = \cos(n\theta) + i\sin n\theta$. Also noting that $e^{in\theta} = (e^{i\theta})^n$. With $n = 2$ and a little algebra, we can easily derive the double and half angle formulas.