Improper Integrals for Math 230
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An improper integral can arise in one of two basic ways. We either have something infinite in our limits of integration or we have the function we are trying to integrate goes to infinity somewhere in the region we are integrating in. (Or possibly both occur). We have to deal with these problems through the use of limits.

Infinity in our Limits of Integration:

If we have an integral of the form $\int_a^\infty f(x)dx$ we simply look at the proper definite integral $\int_a^b f(x)dx$ while taking the limit as b approaches infinity. Saying that $\int f(x)dx = F(x) + C$ we have:

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx = \lim_{b\to\infty} F(b) - F(a)$$

We can similarly deal with the lower limit going to $-\infty$ and if both the upper and lower limits of integration go to infinity (plus and minus respectively) we deal with the integral $\int_a^b f(x)dx$ and take the limits as b goes to infinity and a goes to negative infinity. This means we need **TWO** limits.

Infinity in our Integrand

This occurs when we have a singularity or pole in our function being integrated. This is just the math-fancy way of saying that our function $f(x)$ will "explode" and go to $\pm\infty$. We have the integral $\int_a^b f(x)dx$ the problem is that at $x = c$, $a \le c \le b$, $f(x) = \pm\infty$. We have to break up our original integral into two seperate integrals to deal with this singularity.

$$\int_a^b f(x)dx = \lim_{t\to c^-} \int_a^t f(x)dx + \lim_{s\to c^+} \int_s^b f(x)dx$$

We notice that we can't simply take $\lim_{t\to c}$ but we have to pick whether it approaches c from above or below. If c is our upper limit of integration, we have to approach c from below. If c is our lower limit of integration, we have to approach c from above.

A Few Notes

$\lim_{b\to\infty} \frac{1}{b^p} = 0$ if $p > 0$, $\lim_{b\to\infty} \frac{1}{b^p} = \infty$ if $p < 0$

$\infty = \lim_{b\to 0^+} \frac{1}{b^n} \neq \lim_{b\to 0^-} \frac{1}{b^n} = -\infty$ when $n > 0$

If we do any substitution to solve our integral, we must remember that the limits will change. The limits of the original integral are in terms of x, if we do a u-substitution, we must get our limits in terms of u, if we do a trig substitution, we must get our limits in terms of θ.