Arc Length and Surface Area for Math 230

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Arc Length

Given a curve, we want to determine just how long this curve would be if we were to straighten it out and measure it. Being as we left our string and rulers at home, we see that we can attack this problem through the use of calculus. We can derive our formula fairly easily using the Pythagorean Theorem and the idea of taking the differential along the curve, we arrive at:

$$s = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\ dx$$

Another, often helpful way to think about this is that our ds is our speed and we integrate over the speed to find out how far our curve has travelled.

Similarly, if we are given a function in y and we want to find the arclength, we get:

$$s = \int_a^b \sqrt{1 + (\frac{dx}{dy})^2}\ dy$$

Surface Area of Revolution

Before we try to find surface areas of revolution for difficult surfaces, let us first think of what happens to find the surface area of a nice surface, such as a cylinder. In a cylinder, we find the surface area is the product of the circumference of the base circle, $2\pi r$ and the height at a given point. We now want to generalize this idea to more complicated surfaces. If we have revolved a surface around the y-axis, our radius is the function of the curve we are revolving around and our infinitessimal height is our ds, the differential of the arclength. Since both of these change continuously, we see that we must take an integral to find the surface area. So, for a curve rotated about the x-axis we get:

$$SA = \int_a^b 2\pi x ds = \int_a^b 2\pi x \sqrt{1 + (\frac{dy}{dx})^2}\ dx$$

We see that we can get a curve that generates our surface given in terms of x or y and that it can be rotated about the x-axis or the y-axis. So, we get four different formulas that are just a variation on the same idea.

$$SA = \int_a^b 2\pi r ds$$

Where r is the distance from your curve to the axis you are rotating around and ds is the arclength that is appropriate for your curve.