

Parametric Curves for Math 230

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What is a Parametric Curve?

Let's start with a little story. Will decides he wants to cross Green Street, however, there is also a car that is driving down Green Street. If we plot the paths of both Will and the car, the paths will cross. How can we tell if Will crossed safely?

The idea of parametric curves will help us answer this and other exciting questions. We know that there will only be a problem for Will if his position and the position of the car coincide at the same time. If Will is halfway across the street at time 3 seconds and the car is in that same spot at time 45 seconds, Will is safe.

A parametric curve is defined by two functions of the same variable, which we will call t (We can think of this as time as in the example.) The curves we have encountered so far have generally defined y as a function of x , but now we are going to define y and x as functions of t . We will often see a parametric curve given by its parametric equations:

$$x = f(t), \quad y = g(t)$$

We will take care to note that parametric curves can often be described by different parametric equations. For instance, I can describe the unit circle by $y = \pm \cos t$, $x = \pm \sin t$, or $y = \pm \sin t$, $x = \pm \cos t$ on the interval $0 \leq t \leq 2\pi$.

Special Parametric Curves

Imagine that we take a point on a circle, and mark its path on the $x - y$ plane as we roll the circle along the x -axis. The curve we have just created is called a "cycloid." It can be described parametrically by:

$$x = a(t - \sin t), \quad y = a(1 - \cos t)$$

The Lissajous curves are parametric curves of the form:

$$x = \cos(at), \quad y = \sin(bt)$$

Differentiating Parametric Curves

When we have a parametric curve, a natural question to ask is: how do we find a curve tangent at a point? or how fast is this curve going? The first

question will actually require us to do most of the work towards answering the second. If we want to find a curve tangent to our given curve, $x = f(t)$, $y = g(t)$, we have to find $\frac{dy}{dx}$. We now recall the product rule and use a little algebra to get something we can use. $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Now speed is simply the magnitude of this quantity at a given point, or we can use a little Pythagorean Theorem magic to see that the speed is simply: $\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}$.

Now, if we want to find the second derivative of y with respect to x , that is $\frac{d^2y}{dx^2}$, we find:

$$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}}{dx} = \frac{d\frac{dy}{dx}}{dt}$$

Polar Parametric Curves

A polar equation is generally given by $r = f(\theta)$, now if we want to get a parametric equation from this, we recall that $x = r \cos \theta$, and $y = r \sin \theta$. We then see we can form the parametric equation:

$$x = f(\theta) \cos \theta, \quad y = f(\theta) \sin \theta$$

We can then see that the derivative of this parametric curve will be:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}$$