Integrals with Parametric Curves for Math 230

William Green

Area Under a Parametric Curve

Given a parametric curve defined by $x = f(t)$, and $y = g(t)$, we want to find the area under this curve. We recall that the area under a curve can be found by $A = \int_a^b y dx$, which is easy enough to convert into an intergral in terms of the parametric equations with one catch. The points a and b occur at some values of t, say at $t = \alpha$ and $t = \beta$. Now, we have two cases we must consider. First, if $f(\alpha) < f(\beta)$, then:

$$A = \int_{\alpha}^{\beta} g(t)f^{'}(t)dt$$

Secondly, if $f(\alpha) > f(\beta)$, then:

$$A = \int_{\beta}^{\alpha} g(t)f^{'}(t)dt$$

Arc Length

We remember from our infimitesimal triangle that $ds = \sqrt{dx^2 + dy^2}$, but for a parametric curve defined by $x = f(t)$, and $y = g(t)$, we have to do a little more work. We see that $dx = f^{'}(t)dt$ and $dy = g^{'}(t)dt$, so we have:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt = \sqrt{[f'(t)]^2 + [g'(t)]^2}dt$$

So, the actual length of the arc is given by:

$$s = \int_{\alpha}^{\beta} ds = \int_{\alpha}^{\beta} \sqrt{[f'(t)]^2 + [g'(t)]^2}dt$$

Surface Area

When we revolve a curve around an axis and want to know the surface area of the resulting solid, we can simply substitute what we know about parametric curves into what we know about surface areas of solids of revolution. The surface area of a curve revolved around the x-axis is given by: $SA = \int_{x=a}^{b} 2\pi y ds$. Around the y-axis, we have: $SA = \int_{x=a}^{b} 2\pi x ds$. In parametric integrals, we have:

$$SA = \int_{t=\alpha}^{\beta} 2\pi y ds = \int_{t=\alpha}^{\beta} 2\pi g(t)\sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

Around the y-axis, we have: $SA = \int_{x=a}^{b} 2\pi x ds$. In parametric integrals, we have:

$$SA = \int_{t=\alpha}^{\beta} 2\pi x ds = \int_{t=\alpha}^{\beta} 2\pi f(t)\sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

Volume

Similarly, we can easily translate what we know about parametric curves and volumes of revolution to find fairly nice formulas. The volume of a curve revolved around the x-axis is given by: $V = \int_a^b \pi y^2 dx$. In parametric integrals we have:

$$V = \int_a^b \pi y^2 dx = \int_{t=\alpha}^{\beta} \pi [g(t)]^2 f^{'}(t) dt$$

The volume of a curve revolved around the y-axis is given by: $V = \int_a^b 2\pi x y dx$. In parametric integrals we have:

$$V = \int_a^b 2\pi x y dx = \int_{t=\alpha}^{\beta} 2\pi f(t) \cdot g(t) f^{'}(t) dt$$