Polar Area Computations for Math 230

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Area in Polar Coordinates

When we want to find the area under a curve in cartesian, we partition the area into little tiny, "infinitessimal" rectangles and add up the sums of the areas of all the tiny rectangles, ydx, to get an estimate of the area under the curve. To make our estimate exact, instead of summing, we take the integral $\int ydx$.

We do nearly the same thing to get area enclosed by a curve in polar, but we can't really make rectangles, so we make "infinitessimal" sectors. That is, we partition our curve by dividing the angle θ into tiny pieces, then we take a little piece of the curve, a sector, whose area is $\frac{1}{2}r^2d\theta$, since on the infinitessimal level, we have a triangle whose sides are r and $rd\theta$, hence it has area $\frac{1}{2}r \cdot rd\theta$. and take a sum of all the little sectors to get an approximate area. Again, to make this exact, we make it an integral. So, the area enclosed by a polar curve is:

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

Where, α and β are your limits of integration of the θ variable.

Area Between two Curves

In very similar fashion to finding the area between two curves in cartesian coordinates, we can simply subtract off the area of the "inner" curve from the area of the "outer" curve. So, if I have two conviently named curves in polar form, r_{outer} and r_{inner}, the area between them is:

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_{outer}^2 d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_{inner}^2 d\theta = \frac{1}{2} \int_{\alpha}^{\beta} [r_{outer}^2 - r_{inner}^2] d\theta$$

We must always be careful to be sure that our limits of integration are correct. We can get something very different from the answer we seek if do not have the correct limits of integration. If the two curves do not meet, the limits of integration are determined by the question, that is α and β could be any values between 0 and 2π. However, if the two curves intersect, it will be the points of intersection that determine the limits of integration.