Will's Guide To Life, Final Volume

Math 181

Summer 2008

Preliminaries

A few important things to remember. Remember that the work is more important than the final answer. There is a limit on time, so work hard and work efficiently, do not spend all of your time working any one problem. It is better to have studied too much and be overprepared than to understudy and do poorly. Make sure you know the Review Vocabulary at the end of each chapter.

Section 1.1

Operations research is finding the best method to find the best, or optimal solution. What a graph, path and circuit are. Using a graph to simplify and model a real-world problem. Eulerian Circuits, what they are and for what type of problems they represent optimal solutions.

Section 1.2

Connected graphs, valence of a vertex and Euler's Theorem. How to find Eulerian Circuits, using Euler's Theorem first to make sure one exists.

Section 1.3

The Chinese Postman Problem, what it is and what it models in real life. How to eulerize a graph by reusing existing graphs. Finding the best eulerizations for graphs with a small number of edges. Rectangular networks, using the edge-walker technique. Finding better eulerizations than the edge-walker in non-rectangular graphs.

Section 1.4

What a digraph is and what it can model in real life.

Section 2.1

What a Hamiltonian Circuit is and the types of real world problems it can be used to model. How to construct a graph with no Hamiltonian Circuits. Weighted graphs and minimum-cost Hamiltonian circuits. Algorithms. Complete graphs. The Method of Trees to find Hamiltonian circuits in complete graphs. The Fundamental Principle of Counting. The factorial and the number of Hamiltonian Circuits in a complete graph with n vertices.

Section 2.2

The Traveling Salesman Problem (TSP). Different real world problems to which the TSP is applicable.

Section 2.3

Different algorithms you can use to try to solve the TSP. Greedy algorithms and how they can fail. The Nearest-Neighbor Algorithm, the Sorted-Edges Algorithm. The use of heuristic algorithms and why they can be good to use even if they do not find the optimal solution. Minimum-cost spanning trees and Kruskal's Algorithm. What problems finding a minimum-cost spanning tree is good for.

Section 2.4

The order-requirement digraph and why it can be better than just a digraph. The critical path and how it determines how long a series of tasks will take.

Section 9.1

Preference list ballots and the information it gives you other than just a favorite. Majority Rule, the three properties it satisfies. What it means for a voting system to be monotone. What May's Theorem says about two candidate voting systems. Dictatorship, imposed rule and minority rule and the properties they fail to satisfy. Condorcet's Method for three or more candidates. Condorcet's Voting Paradox and how to construct a situation in which it occurs.

Section 9.2

Plurality voting, what is good and bad about this voting system. The Condorcet Winner Criterion, manipulability. Rank methods, in particular the Borda count. The Borda score, independence of irrelevant alternatives, how one voter can manipulate the Borda count. Sequential pairwise voting, agendas and how this system can fail. The Pareto Condition. The Hare system, Plurality Runofff and what property they fail to satisfy. Where these voting systems are used in the real world.

Section 9.3

Arrow's Impossibility Theorem. What it means for voting systems with three or more candidates. The weak version of Arrow's Impossibility Theorem. Why this doesn't mean we give up on three or more candidate voting systems.

Section 9.4

Approval voting. Where it is used in the real world.

Section 10.1

Strategic voting and manipulation by submitting insincere or disingenuous ballots. How the Borda count can be manipulated by one voter. A unilateral change in ballot. The formal definition of manipulability, it doesn't include changing an election into a tie! The relationship between monotonicity and nonmanipulability. May's Theorem for two candidate elections. Condorcet's Method is nonmanipulable.

Section 10.2

Manipulating the Borda count for more than three candidates. How the Borda Count cannot be manipulated with exactly three candidates. How to construct elections under the Borda count that can be manipulated with any number of voters by starting at base cases and adding pairs of ballots that cancel each other out. Manipulating Runoff systems, Hare and Plurality runoff. Agenda manipulation in sequential pairwise voting. Group manipulability of plurality voting.

Section 10.3

The four properties of Condorcet's Method we would like a voting system to have. The Gibbard-Satterthwaite Theorem for three or more candidate voting systems. How the GS Theorem relates to Arrow's Theorem. The weak version of the GS Theorem. What this means for three or more candidate voting systems.

Section 10.4

The Chair's Paradox. How the chair, having tie-breaking power will not get their choice if all voters act rationally. Strategies and how one can weakly dominate another.

Section 11.1

Weighted Voting Systems. We only use them to answer "yes or no" questions, you can either vote yes for a motion or no, that is there are only two candidates. Examples of how these voting systems arise in real life. The shorthand $[q : w_1, w_2, \ldots, w_n]$ what it means and how it relates to weighted voting systems. Dictators, veto power and dummy voters. A power index as a way to determine a voter's power in a weighted voting system. Permutations and factorials. The pivotal voter, each permutation has exactly one. How to determine a Shapley-Shubik Power Index, it is always a number between 0 and 1. How to compute the S-S power index for a small number of voters. The two tricks for computing with more than 3 voters, same weight means same power and adding up all the power indexes gives exactly 1.

Section 11.2

The Banzhaf Power Index. The three different kinds of coalitions. The critical voter, a coalition does not have to have any and it could have several. The definition of the Banzhaf Power Index. The Extra Votes Principle and how it helps us find critical voters. The winning/blocking duality, why it works and how it helps us find BPIs. Combinations, how many coalitions with k voters from n total voters is given by $C_k^n = \frac{n!}{k!(n-k)!}$. Remember that factorials can cancel out nicely if you have to calculate these! The duality formula and the addition formula for combinations. Pascal's Triangle.

Section 11.3

Comparing voting systems. What it means for two voting systems to be equivalent. Minimal coalitions, coalitions in which every voter is critical. These are important because a voting system can be completely described by its minimal coalitions. The three rules for a list of minimal coalitions. For three voters, there are exactly five weighted voting systems, a dictatorship, consensus rule, a clique, chair veto and majority rule. Know what each system is, how it works and its minimal winning coalitions.

Section 13.1

Fair Division. How do we divide up objects among different parties so that everyone is happy? The adjusted winner procedure. Ideas of "fairness," proportional and envy-free, what these terms mean. How the adjusted winner procedure is done when two parties need to allocate objects. How you use the point ratio to determine allocations. Equitable and Pareto-optimal.

Section 13.2

The Knaster Inheritance Procedure. n parties place bids on items to be allocated. The highest bidder receives the object, but have to pay in $\frac{n-1}{n}$ of their bid, and each other party takes out $\frac{1}{n}$ of their bid. The remaining money is split evenly among all the players. Drawback of this procedure is that for valuable items, players need to have large amounts of cash available. Proportional, but not envy-free.

Section 13.3

Taking Turns. Fair-division procedure for exactly two players. They each take turns choosing items. Three main questions for this procedure: how do we decide who goes first? Should the second chooser be compensated? Is it best to use a sincere strategy when choosing items? The bottom-up strategy using preference lists.

Section 13.4

Divide-and-Choose. One party is the divider and the other is the chooser. Strategies can be used if you know how the other player values the items being chosen. Division is both proportional and envy-free.

Logic: Not in the Book

Some basic logic. A $\Rightarrow$ B is equivalent to Not B $\Rightarrow$ Not A. Not equivalent to B $\Rightarrow$ A or Not A $\Rightarrow$ Not B. A if and only if B means that both A $\Rightarrow$ B and B $\Rightarrow$ A are true.

Section 13.5

Cake-Division Procedures: Proportionality. Definitions of proportionality and envy-free for cake division procedures. For two players proportional if and only if envy-free. Envy-free always implies proportional. For more than two players, proportional is easy to find, envy-free is hard to find. Steinhaus Proportional

Procedure for Three Players (Lone Divider). Banach-Knaster Proportional Procedure for Four or More Players (Last Diminisher).

Section 13.6

Cake-Division Procedures: The Problem of Envy. Steinhaus and Banach-Knaster are both not envy-free. The Selfridge-Conway envy-free procedure. Main Idea: First find an envy-free allocation for part of the cake (all but some trimmings.) Second, divide the trimmings up in an envy-free manner. You should be able to explain this process!

Modular Arithmetic and Divisibility: Not in the Book

Modular arithmetic. The modulus is the number at which we reset to zero, also the number we divide by and take remainders. $a \equiv b \mod n$ means that $a - b$ is a multiple of n. $a|b$ means there is a whole number c such that $b = ca$. Equivalent to b is a multiple of a. How to tell when a number is a multiple of 2, ends in 0, 2, 4, 6, or 8. A number is a multiple of 4 if and only if the last two digits of the number are a multiple of 4. Tricks for determining if a number is a multiple of 3, 9 or 11. Using these tricks to determine if a number is a multiple of 6, 18, 33, 99, etc.