

Math 220 Practice Exam I

Chapters 0 and 1

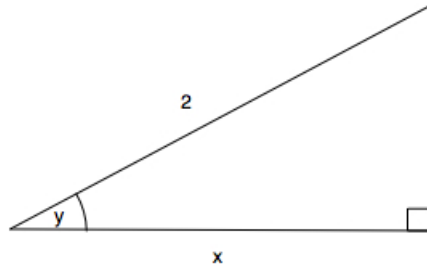
Please alert me of any mistakes in the solutions via email.

Problem 1. a) Precisely define the inverse cosine function.

b) Simplify the following expressions $\sec(\cos^{-1} \frac{x}{2})$ and state clearly for which values of x this simplification holds.

Solution. a) $y = \cos^{-1} x$ if and only if $x = \cos y$ and $0 \leq y \leq \pi$.

b) To do this, we note the definition above and construct a reference triangle. Noting that in a right triangle, $\cos y$ is the adjacent leg divided by the hypotenuse.



Where the second leg of the triangle is $\sqrt{4-x^2}$.

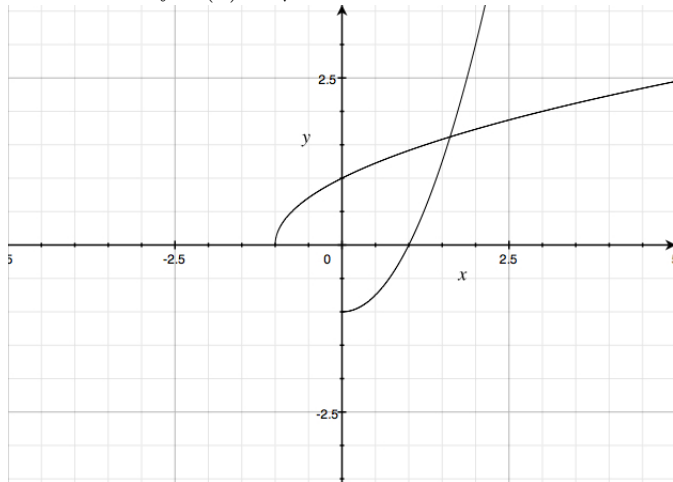
Since secant is the hypotenuse divided by the adjacent leg, $\sec(\cos^{-1} \frac{x}{2}) = \frac{2}{x}$. Clearly we cannot have $x = 0$, and by definition of inverse cosine, we need $-1 \leq \frac{x}{2} \leq 1$. (The easier way to figure out this inequality is that we need the leg $\sqrt{4-x^2}$ to exist.) The values of x for which this simplification hold are $[-2, 0) \cup (0, 2]$.

□

Problem 2. For the function $f(x) = x^2 - 1$, find the inverse function and the restricted domain. Clearly explain why you restricted the domain. Finally, graph both functions on the same axes.

Solution. We need to restrict the domain of $f(x)$ so that it is one-to-one and will hence have an inverse function. If we restrict to the domain $[0, \infty)$, $f(x)$ is one-to-one. We note that the restricted function has range $[-1, \infty)$, which will be the domain of the inverse function. Setting $y = f(x)$ and solving for x , we

will see that $f^{-1}(x) = \sqrt{x-1}$.



□

Problem 3. a) Solve the following equation for x .

$$\ln(2x^3 - 3x^2 + x) - \ln(x) = 0.$$

b) Express the following expression in terms of one natural logarithm

$$2 \log_3 4 - \log_3 \frac{1}{2}.$$

Solution. a) We first use our Rules of Exponents to combine the two logarithms.

$$\ln(2x^3 - 3x^2 + x) - \ln(x) = 0$$

$$\ln\left(\frac{2x^3 - 3x^2 + x}{x}\right) = 0$$

$$\ln(2x^2 - 3x + 1) = 0$$

Now, if we exponentiate both side we have

$$2x^2 - 3x + 1 = 1$$

$$2x^2 - 3x = 0$$

So, $x = 0, \frac{3}{2}$ are our possible solutions. However, we must make sure these answers are in the domain of both logarithmic functions in our original expression. Clearly $x = 0$ cannot be a solution since it is not in the domain of either logarithm. $x = \frac{3}{2}$ is in the domain of both logarithms as $2x^3 - 3x^2 + x, x > 0$ here. So $x = \frac{3}{2}$ is the only solution.

b) Using our rules of logarithms, we can see that

$$\begin{aligned} 2 \log_3 4 - \log_3 \frac{1}{2} &= \log_3(4^2) - \log_3 \frac{1}{2} \\ &= \log_3 \left(\frac{16}{1/2} \right) \\ &= \log_3 32. \end{aligned}$$

Using the Change of Base Formula,

$$2 \log_3 4 - \log_3 \frac{1}{2} = \frac{\ln 32}{\ln 3}.$$

□

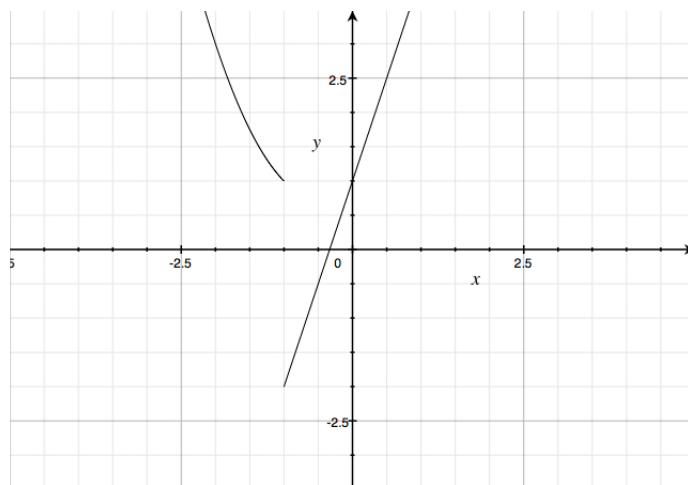
Problem 4. Explain the relationship between the one-sided limits and the existence of the limit.

Solution. A limit exists if and only if the one-sided limits both exist and are equal.

□

Problem 5. Sketch the graph of $f(x) = \begin{cases} x^2 + x + 1 & \text{if } x < -1 \\ 3x + 1 & \text{if } x \geq -1 \end{cases}$, and identify the following limits.

- a) $\lim_{x \rightarrow -1^-} f(x)$.
- b) $\lim_{x \rightarrow -1^+} f(x)$.
- c) $\lim_{x \rightarrow -1} f(x)$.



Solution.

$$\text{a) } \lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} x^2 + x + 1 = 1$$

b) $\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} 3x + 1 = -2$

c) $\lim_{x \rightarrow -1} f(x)$ does not exist since the one-sided limits do not agree.

□

Problem 6. a) State the Squeeze Theorem.

b) Calculate $\lim_{x \rightarrow 0} x^4 \sin\left(\frac{1}{x}\right)$.

Solution. a) Suppose that $f(x) \leq g(x) \leq h(x)$ for all x in an interval (c, d) except possibly at the point $a \in (c, d)$ and that $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$ for some number L . Then, $\lim_{x \rightarrow a} g(x) = L$.

b) Note that $-1 \leq \sin \frac{1}{x} \leq 1$ for all $x \neq 0$. So, using the Squeeze Theorem with $f(x) = -x^4$, $g(x) = x \sin \frac{1}{x}$ and $h(x) = x^4$, on the interval $(-1, 1)$, we can see that $L = 0$, so $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$.

□

Problem 7. Given that $\lim_{x \rightarrow a} f(x) = 4$, $\lim_{x \rightarrow a} g(x) = 2$ and $\lim_{x \rightarrow a} h(x) = -4$. Consider $\lim_{x \rightarrow a} \sqrt{[f(x)]^2 + 2h(x) - 4g(x)}$. Evaluate the limit if possible, or explain why you cannot.

Solution. We cannot evaluate this limit. Since it is a square root, we need $[f(x)]^2 + 2h(x) - 4g(x) \geq 0$ for all x in some interval (c, d) containing a . We do not have this. In particular, we cannot conclude that this limit is zero.

□

Problem 8. Define what it means for a function to be continuous on the interval $[a, b)$.

Solution. A function $f(x)$ is continuous on $[a, b)$ if for all $a < c < b$, $\lim_{x \rightarrow c} f(x) = f(c)$ and $\lim_{x \rightarrow a^+} f(x) = f(a)$.

□

Problem 9. Explain why each function is discontinuous at the given point by indicating which of the conditions to be continuous are not met.

$$f(x) = \begin{cases} x^2 & x < 2 \\ 3 & x = 2 \\ 3x - 2 & x > 2 \end{cases} \quad \text{at } x = 2$$

Solution. Let us consider the one-sided limits.

$$\begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} x^2 = 4 \\ \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} 3x - 2 = 4 \end{aligned}$$

The one-sided limits agree, but are not equal to $f(2)$. Thus, we have a removable discontinuity. □

Problem 10. Compute the following limits.

- a) $\lim_{x \rightarrow \infty} \frac{-3x^3 + 2x^2 + 7x - 8}{2x^3 + 4}$.
 b) $\lim_{x \rightarrow 1} \frac{1}{x^2 - 2x + 1}$.
 c) $\lim_{x \rightarrow 0^+} \tan^{-1}(\ln x)$.

Solution. a) We need to do some algebra to evaluate this limit.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{-3x^3 + 2x^2 + 7x - 8}{2x^3 + 4} &= \lim_{x \rightarrow \infty} \frac{-3x^3 + 2x^2 + 7x - 8}{2x^3 + 4} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \\ &= \lim_{x \rightarrow \infty} \frac{-3 + \frac{2}{x} + \frac{7}{x^2} - \frac{8}{x^3}}{2 + \frac{4}{x^3}} \\ &= \frac{-3}{2} \end{aligned}$$

Where we use the Limit Laws in the last step since the limits of the numerator and denominator both exist and the limit of the denominator is not zero.

- b) Notice that $\frac{1}{x^2 - 2x + 1} = \frac{1}{(x - 1)^2}$.

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{1}{(x - 1)^2} &= \infty \\ \lim_{x \rightarrow 1^-} \frac{1}{(x - 1)^2} &= \infty \end{aligned}$$

Since the one-sided limits agree, we have that $\lim_{x \rightarrow 1} \frac{1}{x^2 - 2x + 1} = \infty$.

- c) We note that $\tan^{-1} x$ is a continuous function on $(-\infty, \infty)$, so

$$\lim_{x \rightarrow 0^+} \tan^{-1}(\ln x) = \tan^{-1} \left(\lim_{x \rightarrow 0^+} \ln x \right).$$

Since $\lim_{x \rightarrow 0^+} \ln x = -\infty$ and $\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}$. So,

$$\lim_{x \rightarrow 0^+} \tan^{-1}(\ln x) = -\frac{\pi}{2}$$

□

Problem 11. Find all asymptotes of the function $f(x) = \frac{x^4}{x^3 + 2}$.

Solution. Since $\lim_{x \rightarrow -\sqrt[3]{2}} x^3 + 2 = 0$ and $\lim_{x \rightarrow -\sqrt[3]{2}} x^4 = 2^{4/3}$, we can show that

$\lim_{x \rightarrow -\sqrt[3]{2}^-} \frac{x^4}{x^3 + 2} = -\infty$ and $\lim_{x \rightarrow -\sqrt[3]{2}^+} \frac{x^4}{x^3 + 2} = \infty$ there is a vertical asymptote at $x = -\sqrt[3]{2}$. (Simply saying that the denominator is zero at this point is NOT a good answer and would not receive many points.)

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{x^4}{x^3 + 2} &= \lim_{x \rightarrow \infty} \frac{x^4}{x^3 + 2} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \\ &= \lim_{x \rightarrow \infty} \frac{x}{1 + \frac{2}{x^3}} = \infty.\end{aligned}$$

Similarly, $\lim_{x \rightarrow -\infty} \frac{x^4}{x^3 + 2} = -\infty$. So there are no horizontal asymptotes.

However, long division shows that $\frac{x^4}{x^3 + 2} = x - \frac{2}{x^3 + 2}$, so there is a slant asymptote that approaches the line $y = x$ as $x \rightarrow \pm\infty$.

□