Math 220 Practice Exam II

Chapter 2

Problem 1. *From the definition of the derivative, calculate* $\frac{d}{dt}(t^3 - 4t^2 + 7)$, *and* $\frac{d}{dx}\frac{1}{\sqrt{2x+1}}$.

Solution. Recall the definition of the derivative of the function $f(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Using the definition, we have

$$\begin{aligned}
\frac{d}{dt}(t^3 - 4t^2 + 7) &= \lim_{h \to 0} \frac{[(t+h)^3 - 4(t+h)^2 + 7] - (t^3 - 4t^2 + 7)}{h} \\
&= \lim_{h \to 0} \frac{[t^3 + 3t^2h + 3th^2 + h^3 - 4(t^2 + 2th + h^2) + 7] - (t^3 - 4t^2 + 7)}{h} \\
&= \lim_{h \to 0} \frac{3t^2h + 3th^2 - 8th + 4h^2}{h} \\
&= \lim_{h \to 0} \frac{h(3t^2 + 3th - 8t + 4h)}{h} \\
&= \lim_{h \to 0} 3t^2 + 3th - 8t + 4h \\
&= 3t^2 - 8t.
\end{aligned}$$

Which agrees with the answer we would get using the Power Rule.

We do the same basic calculation for the other derivative.

$$\begin{aligned}
\frac{d}{dx}\frac{1}{\sqrt{2x+1}} &= \lim_{h\to 0}\frac{\frac{1}{\sqrt{2(x+h)+1}}-\frac{1}{\sqrt{2x+1}}}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{2x+1}-\sqrt{2x+2h+1}}{h(\sqrt{2(x+h)+1})(\sqrt{2x+1})}\\
&= \lim_{h\to 0}\frac{\sqrt{2x+1}-\sqrt{2x+2h+1}}{h(\sqrt{2(x+h)+1})(\sqrt{2x+1})}\cdot\frac{\sqrt{2x+1}+\sqrt{2x+2h+1}}{\sqrt{2x+1}+\sqrt{2x+2h+1}}\\
&= \lim_{h\to 0}\frac{(2x+1)-(2x+2h+1)}{h(\sqrt{2(x+h)+1})(\sqrt{2x+1})(\sqrt{2x+1}+\sqrt{2x+2h+1})}\\
&= \lim_{h\to 0}\frac{-2h}{h(\sqrt{2(x+h)+1})(\sqrt{2x+1})(\sqrt{2x+1}+\sqrt{2x+2h+1})}\\
&= \lim_{h\to 0}\frac{-2}{(\sqrt{2(x+h)+1})(\sqrt{2x+1})(\sqrt{2x+1}+\sqrt{2x+2h+1})}\\
&= \frac{-2}{(\sqrt{2x+1})(\sqrt{2x+1})(\sqrt{2x+1}+\sqrt{2x+1})}\\
&= \frac{-1}{(2x+1)^{3/2}}
\end{aligned}$$

□

Problem 2. *Show that for any real numbers u and v, $|\cos u - \cos v| \leq |u-v|$.*

Solution. The Mean Value Theorem states that if f is continuous on the interval $[u,v]$ and differentiable on (u,v) then there exists a number $c \in (u,v)$ so that

$$f'(c) = \frac{f(u)-f(v)}{u-v}.$$

Since $\cos x$ is continuous and differentiable everywhere, we have it is continuous on $[u,v]$ and differentiable on (u,v). We also know that if $f(x) = \cos x$, then $f'(x) = -\sin x$, and thus $|f'(x)| \leq 1$. Using this inequality and the putting absolute values on conclusion of the Mean Value Theorem, we have

$$\frac{|\cos(u)-\cos(v)|}{|u-v|} \leq 1,$$

which is equivalent to

$$|\cos(u)-\cos(v)| \leq |u-v|.$$

□

Problem 3. *Find the derivative of each function.*
a) $f(x) = \dfrac{4x^2-x+3}{\sqrt{x}}$.

b) $g(w) = \ln(e^{2w^3} + \sin w)$.

c) $h(r) = \dfrac{\csc(r^2)}{r\tan^{-1} r}$.

Solution. a) Here it is tempting to use the Quotient Rule, but it will be much easier to simplify first.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\frac{4x^2 - x + 3}{\sqrt{x}}\right) \\
&= \frac{d}{dx}(4x^{3/2} - x^{1/2} + 3x^{-1/2}) \\
&= 4\left(\frac{3}{2}\right)x^{1/2} - \frac{1}{2}x^{-1/2} + 3\left(\frac{-1}{2}\right)x^{-3/2} \\
&= 6x^{1/2} - \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{-3/2}.
\end{aligned}$$

b) Here we must use the Chain Rule. First note that

$$\frac{d}{dw}e^{2w^3} = e^{2w^3}\frac{d}{dw}(2w^3) = 6w^2e^{2w^3}$$

Now, let $u = e^{2w^3} + \sin w$, so $g(w) = \ln(u)$.

$$\begin{aligned}
\frac{dg}{du}(u) &= \frac{d}{du}\ln u = \frac{1}{u}, \\
\frac{du}{dw} &= \frac{d}{dw}(e^{2w^3} + \sin w) = (6w^2e^{2w^3} + \cos w), \\
g'(w) &= \frac{dg}{du}\frac{du}{dw} = \frac{1}{u}(6w^2e^{2w^3} + \cos w) \\
&= \frac{1}{e^{2w^3} + \sin w}(6w^2e^{2w^3} + \cos w)
\end{aligned}$$

c) Here we must use the Quotient, Product and Chain Rules. To use the Quotient Rule, we have $f(r) = \csc(r^2)$, $g(r) = r\tan^{-1} r$. We will have to use the Chain Rule to find $f'(r)$ and the Product Rule to find $g'(r)$.

$$\begin{aligned}
f'(r) &= -\csc(r^2)\cot(r^2)\frac{d}{dr}r^2 = -2r\csc(r^2)\cot(r^2) \\
g'(r) &= \tan^{-1} r + r\frac{1}{1+r^2}
\end{aligned}$$

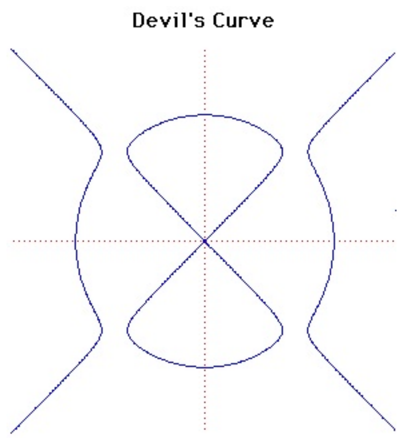
Combining these derivatives via the Quotient Rule, we have

$$\begin{aligned}
h'(r) &= \frac{f'(r)g(r) - f(r)g'(r)}{[g(r)]^2} \\
&= \frac{[-2r\csc(r^2)\cot(r^2)][r\tan^{-1} r] - [\csc(r^2)][\tan^{-1} r + r\frac{1}{1+r^2}]}{[r\tan^{-1} r]^2}.
\end{aligned}$$

□

Problem 4. *The Devil's Curve is a curve defined by the equation*

$$y^4 - x^4 + ay^2 + bx^2 = 0,$$



where a and b are constants. Find the slope of the tangent line to the curve at any point (x, y) on the curve.

Solution. Here we will use Implicit Differentiation. Acting $\frac{d}{dx}$ on both sides of the equation, we have

$$\begin{aligned} \frac{d}{dx}(y^4 - x^4 + ay^2 + bx^2) &= \frac{d}{dx}(0) \\ 4y^3\frac{dy}{dx} - 4x^3 + 2ay\frac{dy}{dx} + 2bx &= 0 \\ \frac{dy}{dx}(4y^3 + 2ay) &= 4x^3 - 2bx \\ \frac{dy}{dx} &= \frac{4x^3 - 2bx}{4y^3 + 2ay}. \end{aligned}$$

So, the slope of the tangent line to the Devil's Curve at an point (x, y) on the curve is given by

$$\frac{4x^3 - 2bx}{4y^3 + 2ay}.$$

□

Problem 5. $f(x) = \dfrac{1}{(4\pi kt)^{1/2}} e^{kx^2/4t}$ *is an important function in analysis of the one-dimensional free Schrödinger equation. Find the first two derivatives of $f(x)$. (Here k is a constant.)*

Solution. Here we need only the Chain Rule for the first derivative.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{1}{(4\pi kt)^{1/2}}e^{kx^2/4t}\right) \\ &= \frac{1}{(4\pi kt)^{1/2}}e^{kx^2/4t}\frac{d}{dx}\left(\frac{kx^2}{4t}\right) \\ &= \frac{2kx}{4t(e\pi kt)^{1/2}}e^{kx^2/4t} \end{aligned}$$

To calculate the second derivative, we will use the Product Rule and the Chain Rule.

$$\begin{aligned} \frac{d^2 f}{dx^2}(x) = \frac{d}{dx}f'(x) &= \frac{d}{dx}\left(\frac{2kx}{4t(e\pi kt)^{1/2}}e^{kx^2/4t}\right) \\ &= \frac{2k}{4t(e\pi kt)^{1/2}}e^{kx^2/4t} + \frac{2kx}{4t(e\pi kt)^{1/2}}e^{kx^2/4t}\frac{d}{dx}\left(\frac{2kx}{4t}\right) \\ &= \frac{2k}{4t(e\pi kt)^{1/2}}e^{kx^2/4t} + \frac{4k^2x}{(4t)^2(e\pi kt)^{1/2}}e^{kx^2/4t} \end{aligned}$$

□

Problem 6. *Find* $f^{(5)}(x)$ *for* $f(x) = 6x^7 - 12x^3 + \cos x$.

Solution. We could compute $f^{(n)}(x)$ for $n \leq 5$, but it is easier to note the following. First, derivatives of cosine repeat themselves after four derivatives, so $\frac{d^5}{dx^5}\cos x = \frac{d}{dx}\cos x = -\sin x$. Second, since derivatives of powers of x are fairly nice, we have

$$\begin{aligned} \frac{d^5}{dx^5}(-12x^3) &= 0, \\ \frac{d^5}{dx^5}(6x^7) &= 6(7\cdot 6\cdot 5\cdot 4\cdot 3)x^{7-5} = 15,120x^2. \end{aligned}$$

So, we can calculate the fifth derivative.

$$f^{(5)}(x) = 15,120x^2 - \sin x$$

□

Problem 7. *a)* $\frac{d}{dt}(t^3 + 2^t)\frac{t^2-1}{t^2+7}$.
b) $\frac{d}{du}\sqrt{u^5 \sec u + \ln u}$.

Solution. a) Here we will use the Product and Quotient Rules. Let $f(t) = t^2 - 1$ and $g(t) = t^2 + 7$ for the Quotient Rule, and let $h(t) = t^3 + 2^t$.

$$\begin{aligned} f'(t) &= 2t \\ g'(t) &= 2t \\ h'(t) &= 3t^2 + 2^t(\ln 2) \end{aligned}$$

If we call $q(t) = \dfrac{f(t)}{g(t)}$, the Product Rule gives

$$\begin{aligned}\frac{d}{dt}(t^3 + 2^t)\frac{t^2-1}{t^2+7} &= h'(t)q(t) + h(t)q'(t) \\ &= h'(t)\frac{f(t)}{g(t)} + h(t)\frac{f'(t)g(t) - f(t)g'(t)}{[g(t)]^2},\end{aligned}$$

where substituted for $q(t)$ and used the Quotient Rule in the second equality. We can now arrive at our final answer.

$$\frac{d}{dt}(t^3 + 2^t)\frac{t^2-1}{t^2+7} = (3t^2 + 2^t(\ln 2))\frac{t^2-1}{t^2+7} + (t^3 + 2^t)\frac{(2t)(t^2+7) - (t^2-1)(2t)}{(t^2+7)^2}$$

□

Problem 8. *Given $f(x)$ has an inverse $g(x)$, find $g'(2)$ if $f(x) = \sqrt{x^3 + 2x + 4}$.*

Solution. Here we need to use the Theorem on derivatives of Inverse Functions. If f is differentiable at x and has an inverse function $g(x)$, then

$$g'(x) = \frac{1}{f'(g(x))},$$

if $f'(g(x)) \neq 0$.

So, we need to find $f'(x)$ and $g(2)$. Recall that for inverses, if $f(a) = 2$, then $f^{-1}(2) = 2$. Since $f(0) = \sqrt{4} = 2$, we have that $g(2) = 0$. We need to use the Chain Rule to find $f'(x)$.

$$\begin{aligned}f'(x) &= \frac{d}{dx}\sqrt{x^3 + 2x + 4} \\ &= \frac{1}{2}(x^3 + 2x + 4)^{-1/2}\frac{d}{dx}(x^3 + 2x + 4) \\ &= \frac{3x^2 + 2}{2}(x^3 + 2x + 4)^{-1/2}\end{aligned}$$

So, we have

$$\begin{aligned}g'(x) &= \frac{1}{f'(g(x))} = \frac{1}{f'(0)} \\ &= \frac{1}{\frac{3(0)^2+2}{2}((0)^3 + 2(0) + 4)^{-1/2}} \\ &= \frac{1}{4^{-1/2}} \\ &= 2\end{aligned}$$

□

Problem 9. *Find* $h'(x)$ *if* $h(x) = \sqrt[3]{3^x + \sqrt{4x + \cos x}}$.

Solution. Here we need to use the Chain Rule. We rewrite the roots as exponents to make calculation clearer.

$$
\begin{aligned}
h'(x) &= \frac{d}{dx}\left(3^x + (4x+\cos x)^{1/2}\right)^{1/3} \\
&= \frac{1}{3}\left(3^x + (4x+\cos x)^{1/2}\right)^{-2/3}\frac{d}{dx}\left(3^x + (4x+\cos x)^{1/2}\right) \\
&= \frac{1}{3}\left(3^x + (4x+\cos x)^{1/2}\right)^{-2/3}\left(3^x(\ln 3) + \frac{1}{2}(4x+\cos x)^{-1/2}\frac{d}{dx}(4x+\cos x)\right) \\
&= \frac{1}{3}\left(3^x + (4x+\cos x)^{1/2}\right)^{-2/3}\left(3^x(\ln 3) + \frac{1}{2}(4x+\cos x)^{-1/2}(4-\sin x)\right)
\end{aligned}
$$

□

Problem 10. *Calculate* $\frac{d}{dx}(x^{4-x^2})$.

Solution. None of the nice rules of differentiation apply here. The function we wish to differentiate is neither a polynomial nor an exponential. Recall Logarithmic Differentiation (a fancy name for cleverly using the Chain Rule.)

$$\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$$

So, if calculating the derivative of the natural logarithm of a function is easier, we use the following identity.

$$f'(x) = f(x)\left(\frac{d}{dx}\ln(f(x))\right)$$

If $f(x) = x^{4-x^2}$, consider $\ln(f(x)) = \ln\left(x^{4-x^2}\right) = (4-x^2)\ln x$.

$$
\begin{aligned}
\frac{d}{dx}\ln(f(x)) &= \frac{d}{dx}\left((4-x^2)\ln x\right) \\
&= (-2x)\ln x + (4-x^2)\frac{1}{x}
\end{aligned}
$$

Using the identity above, we have

$$\frac{d}{dx}(x^{4-x^2}) = \left(x^{4-x^2}\right)\left((-2x)\ln x + (4-x^2)\frac{1}{x}\right).$$

□