Math 220 Practice Exam III

Chapter 3

Problem 1. *a) Calculate the limit* $\lim_{x\to\infty} \frac{\ln x}{e^x}$.

b) The limit in part a) tells us what about the comparative rates at which $\ln x$ *and* e^x *go to infinity? Specifically, which function goes faster?*

c) What can we conclude about $\lim_{x\to\infty} (2e^x - \ln x)$.

Solution. a) We first notice that $\ln x$ is differentiable when $x > 0$ and e^x is differentiable. Second, the limit is in the form “$\frac{\infty}{\infty}$”.

$$\begin{aligned} \lim_{x\to\infty} \frac{\ln x}{e^x} &\overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{e^x} \\ &= \lim_{x\to\infty} \frac{1}{xe^x} = 0 \end{aligned}$$

b) This tells us that e^x goes to infinity much faster than $\ln x$.

c) Since we know e^x goes to infinity much faster than $\ln x$, we will have that $2e^x - \ln x \leq 2e^x - e^x = e^x$ when x is very large, and thus the limit will go to infinity.

$$\lim_{x\to\infty} (2e^x - \ln x) \geq \lim_{x\to\infty} e^x = \infty$$

□

Problem 2. *Determine where the function is increasing or decreasing, concave up or concave down* <u>*and*</u> *classify all extrema and inflection points.*

a) $f(x) = \frac{1}{3}x^3 - 5x^2 + 16x + 3$.

b) $g(x) = \ln(x^2 - 1)$.

Solution. a) We need to calculate the first two derivatives of $f(x)$.

$$\begin{aligned} f'(x) &= x^2 - 10x + 16 = (x-8)(x-2) \\ f''(x) &= 2x - 10 \end{aligned}$$

Notice that $f'(x) = 0$ when $x = 2$ or $x = 8$. These are our critical numbers. We notice that when $x < 2$, $f'(x) > 0$ as both $(x-8), (x-2) < 0$. Further, on $(2, 8)$ we have that $f'(x) < 0$ and on $(8, \infty)$ we have $f'(x) > 0$. So, $f(x)$ is increasing on $(-\infty, 2) \cup (8, \infty)$ and decreasing on $(2, 8)$. The First Derivative Test tells us that $x = 2$ is a local maximum and $x = 8$ is a local minimum. Our picture is as follows.

$$\begin{array}{c|ccccc} f'(x) & + & 0 & - & 0 & + \\ x & \underline{\qquad} & 2 & \underline{\qquad} & 8 & \underline{\qquad} \end{array}$$

The Second derivative is zero at $x = 5$ where it changes signs from negative to positive. Thus $(5, f(5))$ is an inflection point and $f(x)$ is concave down on $(-\infty, 5)$ and concave up on $(5, \infty)$.

b) We need the first two derivatives of $g(x)$.

$$\begin{aligned} g'(x) &= \frac{2x}{x^2-1} \\ g''(x) &= \frac{2(x^2-1) - (2x)(2x)}{(x^2-1)^2} = \frac{-2x^2-1}{(x^2-1)^2} \end{aligned}$$

Notice that $g'(x) = 0$ when $x = 0$, but there are critical points at $x = 0$, $x = -1$ and $x = 1$ since $g'(x)$ is not defined at $x = \pm 1$. An analysis of the signs of $2x$ and $x^2 - 1$ at various points reveals the following.

$$\begin{array}{c|ccccccc} g'(x) & - & DNE & + & 0 & - & DNE & + \\ x & \underline{\qquad} & -1 & \underline{\qquad} & 0 & \underline{\qquad} & 1 & \underline{\qquad} \end{array}$$

However, $g(x)$ is decreasing on $(-\infty, -1)$ and $g(x)$ is increasing on $(1, \infty)$ as it is not defined on $[-1, 1]$.

We note that $g''(x) \neq 0$ as it is always negative where it is defined. Thus, $g(x)$ is concave down on $(-\infty, -1) \cup (1, \infty)$, and there are no inflection points.

□

Problem 3. *Use a linear approximation to estimate the following numbers. Be sure to state the functions and value x_0 you employ.*

a) $\sin(3.1)$.

b) $\sqrt[4]{80.55}$.

c) $\ln(3)$.

Solution. a) Here we are making a linear approximation to the function $f(x) = \sin x$ about the point $x_0 = \pi$.

$$\begin{aligned} L(x) &= f(x_0) + f'(x_0)(x - x_0) \\ &= 0 - 1(3.1 - \pi) \approx .04159 \end{aligned}$$

b) Here we are making an approximation to the function $f(x) = x^{1/4}$ about the point $x_0 = 81$.

$$\begin{aligned} L(x) &= f(x_0) + f'(x_0)(x - x_0) \\ &= 3 + \frac{1}{4}(81)^{-3/4}(80.55 - 81) \approx 2.9958 \end{aligned}$$

c) Here we are making an approximation to the function $f(x) = \ln(x)$ about the point $x_0 = e$.

$$\begin{aligned} L(x) &= f(x_0) + f'(x_0)(x - x_0) \\ &= 1 + \frac{1}{e}(3 - e) = \frac{3}{e} \approx 1.1036 \end{aligned}$$

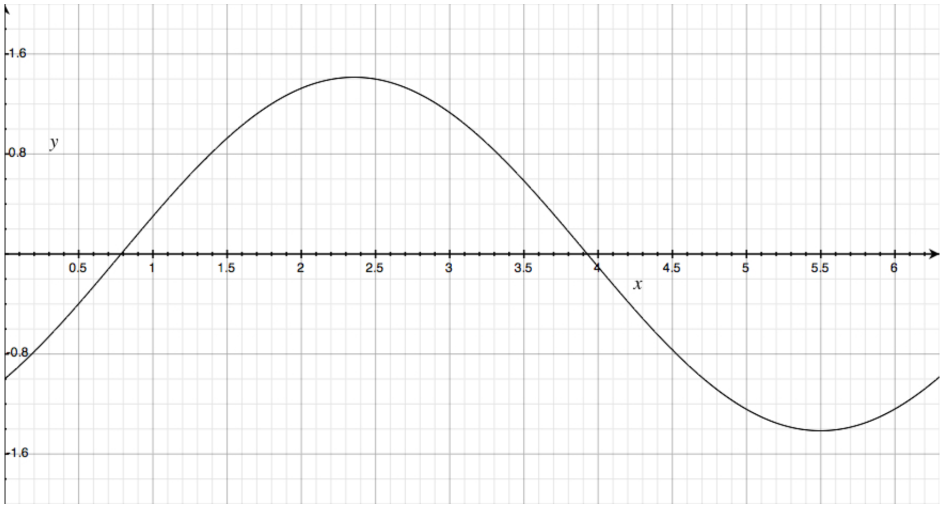
□

Problem 4. *Determine all significant features (as enumerated in class) and sketch the graph of* $f(x) = \sin x - \cos x$ *on* $[0, 2\pi]$.

Solution. We have seven things to check.

1. We are only concerned with the domain of f on $[0, 2\pi]$, as $\sin x$ and $\cos x$ are defined everywhere, the domain is $[0, 2\pi]$.
2. We know that $f(x)$ is continuous, so there are no vertical asymptotes.
3. $f'(x) = \cos x + \sin x = 0$ when $x = 0$ and $x = \frac{3\pi}{4}, \frac{7\pi}{4}$. We note that $f'(x) > 0$, that is f is increasing, on $[0, \frac{3\pi}{4}) \cup (\frac{7\pi}{4}, 2\pi]$, and $f'(x) < 0$, that is f is decreasing on $(\frac{3\pi}{4}, \frac{7\pi}{4})$.
4. $f'(x)$ is continuous and thus is defined at all points in the domain.
5. $f''(x) = -\sin x + \cos x = 0$ when $x = \frac{\pi}{4}, \frac{5\pi}{4}$. Also, $f''(x) < 0$, that is f is concave down, on $(\frac{\pi}{4}, \frac{5\pi}{4})$, and $f''(x) > 0$, f is concave up n $[0, \frac{\pi}{4}) \cup (\frac{5\pi}{4}, 2\pi]$.
6. There are no horizontal asymptotes since we are only graphing this function on $[0, 2\pi]$. (In fact there are no horizontal asymptotes because this is a periodic function.)
7. $f(0) = -1$, so $(0, -1)$ is the y-intercept. Solving $f(x) = 0$ yields that $x = \frac{\pi}{4}, \frac{5\pi}{4}$ are the x-intercepts.

Combining all this information, we can come up with a plausible sketch.



$\square$

Problem 5. *Suppose that you are blowing up a spherical balloon by adding air at a rate of* $\pi\,\frac{in^3}{s}$. *What is the rate at which the radius changing when the radius of the balloon is* $\frac{32}{3}\pi\, in^3$.

Solution. The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$, where r is the radius. Acting a derivative with respect to time on each side of the equation yields

$$\begin{aligned}\frac{d}{dt}(V) &= \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)\\ \frac{dV}{dt} &= 4\pi r^2\frac{dr}{dt}.\end{aligned}$$

We know that $\frac{dV}{dt} = \pi\frac{cm^3}{s}$ and we want $\frac{dr}{dt}$ when $r = \frac{32}{3}\pi\, cm^3$. Plugging these into our related rates equation,

$$\pi\frac{cm^3}{s} = 4\pi\left(\frac{32}{3}\pi\, cm\right)^2\frac{dr}{dt}.$$

Thus, we have $\frac{dr}{dt} = \frac{1}{4(32\pi/3)^2}\frac{cm}{s}$.

$\square$

Problem 6. *Find the point on the curve* $y = x^2$ *closest to the point* $(3, 4)$.

Solution. The distance between a point on the curve and $(3,4)$ is given by $d(x) = \sqrt{(x-3)^2+(y-4)^2} = \sqrt{(x-3)^2+(x^2-4)^2}$. To minimize this, we will need the derivative.

$$\begin{aligned} d'(x) &= \frac{1}{2\sqrt{(x-3)^2+(x^2-4)^2}}\left[2(x-3)+2(x^2-4)(2x)\right] \\ &= \frac{8x^3-6x-6}{2\sqrt{(x-3)^2+(x^2-4)^2}} \end{aligned}$$

We see that $d'(x) = 0$ precisely when $8x^3-6x-6 = 0$. In fact, it would be easier to note that $d(x)$ is minimized when $[d(x)]^2$ is minimized since $d(x)$ is never negative. (Compare the derivative of $[d(x)]^2$ with what we are setting equal to zero.)

This occurs when $x \approx 1.1776$. When $x < 1.1776$, $f'(x) < 0$ and when $x > 1.1766$, $f'(x) > 0$, so $x = 1.1776$ is the minimum of $d(x)$. Thus, $(1.1766, 1.3844)$ is the closest point on the curve to $(3,4)$.

$\square$

Problem 7. *Suppose a 3 foot long wire is to be cut into two pieces, each of which will be formed into a square. Find the size of each piece to maximize the total area of the two squares.*

Solution. The two pieces of wire will have lengths x and $3-x$. Each piece of wire will be formed into a square with side length $\frac{x}{4}$ and $\frac{3}{4}-\frac{x}{4}$. The total area of the squares is given by

$$\begin{aligned} A(x) &= \left(\frac{x}{4}\right)^2 + \left(\frac{3}{4}-\frac{x}{4}\right)^2 \\ &= \frac{9}{16} - \frac{3}{8}x + \frac{x^2}{8} \end{aligned}$$

To maximize this, we note that $0 \le x \le 3$ and we need to find the absolute maximum of $A(x)$ on $[0,3]$. We note that

$$\begin{aligned} A(0) &= \frac{9}{16} \\ A(3) &= \frac{9}{16} - \frac{3}{8}(3) + \frac{(3)^2}{8} = \frac{9}{16} \end{aligned}$$

We need to test any critical numbers that occur on $[0,3]$.

$$\frac{dA}{dx}(x) = -\frac{3}{8} + \frac{x}{4}$$

Setting this equal to zero, we see that $x = \frac{3}{2}$ is a critical number. As $A'(x) < 0$ when $x < \frac{3}{2}$ and $A'(x) > 0$ when $x > \frac{3}{2}$, this critical point is a minimum.

$$A\left(\frac{3}{2}\right) = \frac{9}{16} - \frac{3}{8}\left(\frac{3}{2}\right) + \frac{(3/2)^2}{8} = \frac{9}{32}$$

So, the maxima occur when $x = 0$ or $x = 3$. Physically, this means the maxima occur when we do not cut the wire.

$\square$

Problem 8. *Calculate the limit:*

$$\lim_{x\to\infty} \frac{\sqrt[3]{x^2+7}}{\sqrt[3]{x^2-4}}.$$

Solution. We first notice that, this problem is easier if we have not learned L'Hôpital's Rule, and are not tempted to use it.

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt[3]{x^2+7}}{\sqrt[3]{x^2-4}} &= \lim_{x\to\infty} \frac{\sqrt[3]{x^2+7}}{\sqrt[3]{x^2-4}} \cdot \left(\frac{1/x^{2/3}}{1/x^{2/3}}\right) \\ &= \lim_{x\to\infty} \frac{\sqrt[3]{1+7x^{-2}}}{\sqrt[3]{1-4x^{-2}}} \\ &= 1\end{aligned}$$

Where all we needed was the normal Limit Laws.

If we use L'Hôpital's Rule here, we note that $\sqrt[3]{x^2+7}$ and $\sqrt[3]{x^2-4}$ are differentiable and this is in the indeterminant form "$\frac{\infty}{\infty}$".

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt[3]{x^2+7}}{\sqrt[3]{x^2-4}} &\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{3}(x^2+7)^{-2/3}(2x)}{\frac{1}{3}(x^2-4)^{-2/3}(2x)} \\ &= \lim_{x\to\infty} \frac{(x^2-4)^{2/3}}{(x^2+7)^{2/3}}\end{aligned}$$

Here we note that $(x^2-4)^{2/3}$ and $(x^2+7)^{2/3}$ are differentiable and this is in the indeterminant form "$\frac{\infty}{\infty}$".

$$\begin{aligned}\lim_{x\to\infty} \frac{(x^2-4)^{2/3}}{(x^2+7)^{2/3}} &\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{2}{3}(x^2-4)^{-1/3}(2x)}{\frac{2}{3}(x^2+7)^{-1/3}(2x)} \\ &= \lim_{x\to\infty} \frac{\sqrt[3]{x^2+7}}{\sqrt[3]{x^2-4}}\end{aligned}$$

So, L'Hôpital's Rule will bring us back to the original limit.

$\square$

Problem 9. *Find all the inflections points of* $\tan^{-1}(x^2)$. *Be sure to justify your answer.*

Solution. Let $f(x) = \tan^{-1}(x^2)$. The inflection points occur when $f''(x)$ changes signs.

$$\begin{aligned} f'(x) &= \frac{1}{1+(x^2)^2}(2x) = \frac{2x}{1+x^4} \\ f''(x) &= \frac{2(1+x^4)-(2x)(4x^3)}{(1+x^4)^2} = \frac{-6x^4+2}{(1+x^4)^2} \end{aligned}$$

If we set the second derivative equal to zero, noting that the denominator is always positive and non-zero, we want to find when $-6x^4+2=0$. Or, when $x^4 = \frac{1}{3}$. That is when $x = \pm\frac{1}{\sqrt[4]{3}}$.

$$\begin{array}{c|ccccc} f''(x) & - & 0 & + & 0 & - \\ x & \rule{2cm}{0.4pt} & -\frac{1}{\sqrt[4]{3}} & \rule{2cm}{0.4pt} & \frac{1}{\sqrt[4]{3}} & \rule{2cm}{0.4pt} \end{array}$$

So, both $\left(\frac{1}{\sqrt[4]{3}}, f\left(\frac{1}{\sqrt[4]{3}}\right)\right)$ and $\left(-\frac{1}{\sqrt[4]{3}}, f\left(\frac{1}{\sqrt[4]{3}}\right)\right)$ are inflection points of $f(x)$.

$\square$