Math 220 Practice Exam IV

Chapter 4

Problem 1. *Find all functions that satisfy the given conditions.*
a) $f''(x) = \cos x - 2\sin x + 24x$.
b) $f'(x) = \dfrac{4}{\sqrt{1-x^2}} - 3\csc x \cot x + e^x$.

Solution. a) We need to anti-differentiate $f''(x)$ twice to obtain $f(x)$.

$$f'(x) = \int (\cos x - 2\sin x + 24x)\, dx$$
$$f'(x) = \sin x + 2\cos x + 12x^2 + C_1$$

So,

$$f(x) = \int \left(\sin x + 2\cos x + 12x^2 + C_1\right)\, dx$$
$$= -\cos x + 2\sin x + 4x^3 + C_1 x + C_2.$$

Here C_1 and C_2 are arbitrary constants of integration.

b)

$$f(x) = \int \left(\frac{4}{\sqrt{1-x^2}} - 3\csc x \cot x + e^x\right)\, dx$$
$$= 4\sin^{-1} x + 3\csc x + e^x + C$$

□

Problem 2. *Compute the following sums.*
a) $\sum_{i=1}^{53}(4i^2 - 5i - 8)$.
b) $\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 - 2\left(\frac{i}{n}\right) + 7\right]$.

Solution. a) First, we let $n = 53$,

$$\sum_{i=1}^{n}(4i^2 - 5i - 8) = 4\sum_{i=1}^{n} i^2 - 5\sum_{i=1}^{n} i - \sum_{i=1}^{n} 8$$
$$= 4\frac{n(n+1)(2n+1)}{6} - 5\frac{n(n+1)}{2} - 8n.$$

Plugging in $n = 53$, we have

$$\sum_{i=1}^{53}(4i^2 - 5i - 8) = 4\frac{53(54)(107)}{6} - 5\frac{53(54)}{2} - 8(53).$$

b)

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 - 2\left(\frac{i}{n}\right) + 7\right] &= \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2 - 2\left(\frac{i}{n}\right) + 7\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2\right] - 2\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)\right] + \frac{1}{n}\sum_{i=1}^{n}7 \\
&= \frac{1}{n^3}\sum_{i=1}^{n}i^2 - \frac{2}{n^2}\sum_{i=1}^{n}i + \frac{1}{n}\sum_{i=1}^{n}7
\end{aligned}$$

Now, we can use of summation identities to get that

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 - 2\left(\frac{i}{n}\right) + 7\right] &= \frac{n(n+1)(2n+1)}{6n^3} - \frac{2n(n+1)}{2n^2} + \frac{7n}{n} \\
&= \frac{2n^2+3n+1}{6n^2} - \frac{n+1}{n} + 7.
\end{aligned}$$

□

Problem 3. *The sum in part b) of the previous problem can be identified as a Riemann Sum.*
a) It represents an approximation of what function on what interval?
b) Calculate the limit as $n \to \infty$ of the sum in part b) of the previous problem.

Solution. a) This represents a Riemann Sum of n rectangles for the function $f(x) = x^2 - 2x + 7$ on the interval $[0, 1]$. The $\frac{1}{n}$ multiplying every term in the initial sum is the Δx. This particular sum is using right endpoints for c_i.

b) This limit will give us the area under the curve $y = f(x)$ on $[0, 1]$.

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 - 2\left(\frac{i}{n}\right) + 7\right] &= \lim_{n\to\infty}\left[\frac{2n^2+3n+1}{6n^2} - \frac{n+1}{n} + 7\right] \\
&= \frac{2}{6} - 1 + 7 = \frac{19}{3}.
\end{aligned}$$

□

Problem 4. *From the definition of area under the curve, find the area under the curve $y = x^2 + 1$ on the interval $[0, 2]$.*

Solution. We need to remember that the definition of area under the curve for a continuous function f on $[a, b]$ is given by

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $\Delta x = \dfrac{b-a}{n}$ and x_i is any point in the i^{th} subinterval.

So, we need to set up this Riemann Sum. Here $\Delta x = \dfrac{2}{n}$, using right end-points we have $x_i = i\dfrac{2}{n}$. We first find A_n.

$$\begin{aligned}
A_n &= \sum_{i=1}^{n} \left[f\left(\frac{2i}{n}\right) \frac{2}{n} \right] \\
&= \frac{2}{n} \sum_{i=1}^{n} \left[\left(\frac{2i}{n}\right)^2 + 1 \right] \\
&= \frac{2}{n} \sum_{i=1}^{n} \frac{4i^2}{n^2} + \frac{2}{n} \sum_{i=1}^{n} 1 \\
&= \frac{8}{n^3} \sum_{i=1}^{n} i^2 + \frac{2}{n}(n) \\
&= \frac{8n(n+1)(2n+1)}{6n^3} + 2
\end{aligned}$$

We now take the limit as $n \to \infty$ to find the area.

$$\begin{aligned}
A &= \lim_{n\to\infty} A_n \\
&= \lim_{n\to\infty} \frac{8n(n+1)(2n+1)}{6n^3} + 2 \\
&= \frac{8}{3} + 2
\end{aligned}$$

This matches the answer we would get using the Fundamental Theorem of Calculus.

□

Problem 5. *The height of a river is given by the function* $h(t) = 4\cos(2\pi t) - 2\sin(2\pi t) + \frac{1}{16}t^3 - \frac{1}{4}t^4$, *with* t *measured in hours. What is the average height of the river from* $t = 0$ *to* $t = 12$*?*

Solution. We want the average value of $h(t)$ on the interval $[0, 12]$. We first need

$$\int_0^{12} \left(4\cos(2\pi t) - 2\sin(2\pi t) + \frac{1}{16}t^3 - \frac{1}{4}t^4 \right) dt$$

The last two terms are easy enough to deal with, but for the first two, we need to use a substitution. Let $u = 2\pi t$, then $du = 2\pi\, dt$.

$$\begin{aligned}\int (4\cos(2\pi t) - 2\sin(2\pi t))\, dt &= \int (4\cos u - 2\sin u)\frac{du}{2\pi} \\ &= \frac{2}{\pi}\sin u + \frac{1}{\pi}\cos u + C \\ &= \frac{2}{\pi}\sin(2\pi t) + \frac{1}{\pi}\cos(2\pi t) + C\end{aligned}$$

By the Fundamental Theorem of Calculus,

$$\int_0^{12}\left(4\cos(2\pi t) - 2\sin(2\pi t) + \frac{1}{16}t^3 - \frac{1}{4}t^4\right) dt$$

$$\overset{FTC}{=} \left[\frac{2}{\pi}\sin(2\pi t) + \frac{1}{\pi}\cos(2\pi t) + \frac{1}{64}t^4 - \frac{1}{20}t^5\right]\Bigg|_0^{12}.$$

The average value is thus

$$\frac{1}{12}\left[\frac{2}{\pi}\sin(2\pi t) + \frac{1}{\pi}\cos(2\pi t) + \frac{1}{64}t^4 - \frac{1}{20}t^5\right]\Bigg|_0^{12}.$$

□

Problem 6. *Consider the function defined by* $\displaystyle\int_{x^3-3x+5}^{\sin x+14x}\left(e^{t^2}\right) dt$.

a) *This is a function of what variable?*
b) *Compute the derivative.*

Solution. a) This is a function of x. t is just a dummy variable.

b) Let us call $F(x) = \displaystyle\int_{x^3-3x+5}^{\sin x+14x}\left(e^{t^2}\right) dt$, as e^{t^2} is continuous, the Fundamental Theorem of calculus tells us that we can differentiate this. However, we need to use the Chain Rule.

$$\begin{aligned}\frac{d}{dx}F(x) &= \frac{d}{dx}\left[\int_{u(x)}^{v(x)}\left(e^{t^2}\right) dt\right] \\ &= \frac{d}{dx}\left[\int_{u(x)}^{c}\left(e^{t^2}\right) dt\right] + \frac{d}{dx}\left[\int_{c}^{v(x)}\left(e^{t^2}\right) dt\right]\end{aligned}$$

where c is any number.

$$\begin{aligned}\frac{d}{dx}F(x) &= \frac{d}{du}\left[\int_u^c\left(e^{t^2}\right) dt\right]u'(x) + \frac{d}{dv}\left[\int_c^v\left(e^{t^2}\right) dt\right]v'(x) \\ &= -e^{u^2}u'(x) + e^{v^2}v'(x) \\ &= -e^{(x^2-3x+5)^2}(2x-3) + e^{(\sin x+14x)^2}(\cos x + 14)\end{aligned}$$

□

Problem 7. *Consider the integral* $\int \cos x \sin x \, dx$.

a) Integrate by substitution with $u = \sin x$.
b) Integrate by substitution with $u = \cos x$.
c) Explain what just happened with your answers to the two previous questions.

Solution. a) If $u = \sin x$, then $du = \cos x \, dx$.

$$\begin{aligned} \int \cos x \sin x \, dx &= \int u \, du \\ &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}\sin^2 x + C \end{aligned}$$

b) If $u = \cos x$, then $du = -\sin x \, dx$.

$$\begin{aligned} \int \cos x \sin x \, dx &= -\int u \, du \\ &= -\frac{1}{2}u^2 + C \\ &= -\frac{1}{2}\cos^2 x + C \end{aligned}$$

c) These are both correct answers. Recall the Pythagorean Identity, $\sin^2 x + \cos^2 x = 1$, or $\sin^2 x = 1 - \cos^2 x$. Plugging this into the answer from part a), we have

$$\begin{aligned} \int \cos x \sin x \, dx &= \frac{1}{2}\sin^2 x + C \\ &= \frac{1}{2}(1 - \cos^2 x) + C \\ &= -\frac{1}{2}\cos^2 x + \frac{1}{2} + C \end{aligned}$$

This is the same answer as in part b) since C is an arbitrary constant and an arbitrary constant plus $\frac{1}{2}$ is still an arbitrary constant.

□

Problem 8. *Evaluate the following.*

a) $\displaystyle\int_1^2 \frac{1}{\sqrt{x}(\sqrt{x}+1)} \, dx$.

b) $\displaystyle\int e^{\tan t} \sec^2 t \, dt$.

c) $\displaystyle\int_1^4 \frac{x^3 + 7x}{x^3} \, dx$.

Solution. a) To calculate this definite integral, we need to use a substitution. Choosing $u = \sqrt{x} + 1$, we have $du = \dfrac{1}{2\sqrt{x}}\,dx$. We first calculate the indefinite integral to use the Fundamental Theorem of Calculus.

$$\begin{aligned}
\int \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx &= \int \left(\frac{2}{\sqrt{x}+1}\right)\left(\frac{1}{2\sqrt{x}}\,dx\right) \\
&= \int \frac{2}{u}\,du \\
&= 2\ln|u| + C \\
&= 2\ln|\sqrt{x}+1| + C
\end{aligned}$$

Now, as the integrand is continuous on $[1, 2]$, the FTC applies and

$$\int_1^2 \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx \overset{FTC}{=} \left(2\ln|\sqrt{x}+1|\right)\Big|_1^2$$

b) This indefinite integral (or anti-derivative) can be calculated by a substitution. Let $u = \tan t$, then $du = \sec^2 t\,dt$.

$$\begin{aligned}
\int e^{\tan t}\sec^2 t\,dt &= \int e^u\,du \\
&= e^u + C \\
&= e^{\tan t} + C
\end{aligned}$$

c) This definite integral can be computed nicely.

$$\begin{aligned}
\int_1^4 \frac{x^3+7x}{x^3}\,dx &= \int_1^4 \left(1 + \frac{7}{x^2}\right)dx \\
&\overset{FTC}{=} \left(x - 7x^{-1}\right)\Big|_1^4
\end{aligned}$$

□

Problem 9. *It is a fact that e^{-x^2} has no antiderivative that can be defined in terms of elementary functions. Explain one method by which we can find* $\displaystyle\int_0^1 e^{-x^2}\,dx$.

Solution. The only way to calculate this is by the definition of area under the curve, that is by calculating a limit of Riemann Sums. We can improve the speed at which our Riemann Sum converges to the actual area under the curve by using one of the following rules.

1. Simpson's Rule
2. Trapezoidal Rule

3. Midpoint Rule

□

Problem 10. *Evaluate the sum* $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^6}{n^7}$.

(Hint: You do not have a formula for $\sum i^6$*. You must use some other method.)*

Solution. As stated in the hint, we do not have a nice formula for this. The only way we can evaluate such a sum is to view the limit of this sum as the area under a curve. That is we need to determine a function f and appropriate choices of c_i and Δx so that

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^6}{n^7} &= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x \\ &= \int_a^b f(x)\,dx \end{aligned}$$

We know that $\Delta x = \frac{b-a}{n}$.

$$\begin{aligned} \frac{i^6}{n^7} &= \left(\frac{i^6}{n^6}\right)\frac{1}{n} \\ &= \left(\frac{i}{n}\right)^6 \frac{1}{n} \end{aligned}$$

This looks like using right endpoints in a Riemann Sum of $f(x) = x^6$ on $[0, 1]$. As f is continuous, it is integrable on $[0, 1]$.

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^6}{n^7} &= \int_0^1 x^6\,dx \\ &\overset{FTC}{=} \frac{1}{7}x^7 \Big|_0^1 \\ &= \frac{1}{7} \end{aligned}$$

□