“Odd Derivative” at Review Session

Math 220 Spring ’09

At the review session, someone asked to calculate the derivative of

$$f(x) = x^{1/\ln(x)}$$

Applying Logarithmic Differentiation, we look at

$$\begin{aligned}\frac{d}{dx}[\ln f(x)] &= \frac{d}{dx}\left[\frac{1}{\ln x}\ln x\right]\\ &= \frac{d}{dx}[1]\\ &= 0\end{aligned}$$

So, applying Logarithmic Differentiation, shows

$$\begin{aligned}f'(x) &= f(x)\frac{d}{dx}\left[\ln(f(x))\right]\\ &= x^{1/\ln(x)}[0] = 0\end{aligned}$$

This would tell us that $f(x)$ is actually a constant function. This looks odd, but is actually true.

<u>**Fact:**</u> $x^{1/\ln x} = e$ when $x > 0$, $x \neq 1$, that is where it is defined.

How we can see this:

$$\begin{aligned}\left(x^{1/\ln x}\right)^{\ln x} &= x^{(1/\ln x)(\ln x)}\\ &= x^1 = x\end{aligned}$$

and

$$e^{\ln x} = x.$$

So, raising $f(x)$ to the power $\ln x$ yields exactly the same thing as raising e to the same power. Similarly, if we take logarithms, we showed above in calculating the logarithmic derivative

$$\ln\left(x^{1/\ln x}\right) = 1 = \ln e.$$

So, it is true that

$$x^{1/\ln x} = e$$

where it is defined. This suggests that $f(x)$ should be the constant function. Recall the change of base formula,

$$a^b = e^{b \ln a}.$$

So,

$$x^{1/\ln x} = e^{(1/\ln x) \ln x} = e^1 = e.$$

So $f(x)$ is actually the constant function.