# Trigonometric Identities

Math 220 Spring '09

There are several important trigonometric identities that are useful in calculus.

## Pythagorean Identities

The classic Pythagorean Identity on a right triangle says $a^2 + b^2 = c^2$ where $a$ and $b$ are the legs of a right triangle and $c$ is the hypotenuse. If we think of a right triangle on the unit circle with $a = \sin\theta$, $b = \cos\theta$ and $c = 1$, we have

$$\sin^2\theta + \cos^2\theta = 1. \tag{1}$$

If we divide both sides of (1) by $\sin^2\theta$, we get one of the other Pythagorean Identities.

$$1 + \cot^2\theta = \csc^2\theta.$$

Similarly, if we divide (1) by $\cos^2\theta$, we get

$$\tan^2\theta + 1 = \sec^2\theta.$$

These three formulas are collectively referred to as the "Pythagorean Identities."

## Angle Addition Formulas

Using Euler's Formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

and properties of exponentials, we can derive the angle addition formulas.

$$\begin{aligned} e^{i(\alpha+\beta)} &= e^{i\alpha}e^{i\beta} \\ &= [\cos(\alpha) + i\sin(\alpha)][\cos(\beta) + i\sin(\beta)] \\ &= [\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)] + i[\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)] \qquad (2) \end{aligned}$$

Since

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta), \tag{3}$$

equating the real and imaginary parts of (2) and (3), we have the angle addition formulas.

$$\begin{aligned} \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\ \sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \end{aligned}$$

## Half and Double Angle Formulas

We can obtain the Double Angle Formulas from taking $\alpha = \beta$ in the angle addition formula.

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$
$$\sin(2\alpha) = 2\cos(\alpha)\sin(\alpha)$$

To derive the half-angle formula, let $\alpha = \dfrac{\beta}{2}$, and consider the following.

$$\begin{aligned}\cos(\beta) &= \cos^2\left(\frac{\beta}{2}\right) - \sin^2\left(\frac{\beta}{2}\right) \\ &= \cos^2\left(\frac{\beta}{2}\right) - \left[1 - \cos^2\left(\frac{\beta}{2}\right)\right]\end{aligned}$$

Solving for cosine, we have

$$\cos^2\left(\frac{\beta}{2}\right) = \frac{1}{2} + \frac{1}{2}\cos(\beta).$$

Similarly, we could have replaced the cosine squared and arrived at

$$\sin^2\left(\frac{\beta}{2}\right) = \frac{1}{2} - \frac{1}{2}\sin(\beta).$$