Improper Integrals Homework 1

Section 6.6

September 12, 2008

Problem 1. *6.6 #8.*

$$\int_0^1 x^{-4/3}\,dx$$

Solution. This integral is improper at $x = 0$ since $\lim\limits_{x\to 0} x^{-4/3}$ does not exist. So, we must use a limit as follows.

$$\begin{aligned}\int_0^1 x^{-4/3}\,dx &= \lim_{a\to 0^+} \int_a^1 x^{-4/3}\,dx \\ &= \lim_{a\to 0^+} -3x^{-1/3}\Big|_a^1 \\ &= \lim_{a\to 0^+} -3 + 3a^{-1/3}\end{aligned}$$

This integral diverges to infinity since $\lim\limits_{a\to 0^+} a^{-1/3} = \infty$.

□

Problem 2. *6.6 #20.*

$$\int_{-\infty}^0 xe^{-4x}\,dx$$

Solution. If we ignore the limits of integration for a minute, and just focus on finding the anti-derivative, we see that this is an integration by parts problem. Picking $u = x$ and $dv = e^{-4x}\,dx$ gives us that $du = dx$ and $v = -\frac{1}{4}e^{-4x}$.

$$\begin{aligned}\int xe^{-4x}\,dx &= -\frac{1}{4}xe^{-4x} - \left(-\frac{1}{4}\right)\int e^{-4x}\,dx \\ &= -\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x} + C \\ &= -\frac{1}{4}e^{-4x}\left(x + \frac{1}{4}\right) + C\end{aligned}$$

Where we factored out the $-\frac{1}{4}e^{-4x}$ since it will make finding the definite integral easier. Now, using this, we can find the answer to the initial problem.

$$\begin{aligned}
\int_{-\infty}^{0} xe^{-4x}\,dx &= \lim_{R\to-\infty}\int_{R}^{0} xe^{-4x}\,dx \\
&= \lim_{R\to-\infty} -\frac{1}{4}e^{-4x}\left(x+\frac{1}{4}\right)\Big|_{R}^{0} \\
&= -\frac{1}{4}e^{-4(0)}\left(0+\frac{1}{4}\right) - \lim_{R\to-\infty} -\frac{1}{4}e^{-4R}\left(R+\frac{1}{4}\right) \\
&= -\frac{1}{16} + \lim_{R\to-\infty} -\frac{1}{4}e^{-4R}\left(R+\frac{1}{4}\right)
\end{aligned}$$

This diverges to $-\infty$ since $R+\frac{1}{4}>0$ and $-\frac{1}{4}e^{-4R}<0$.

A slight variation of this problem is good practice.

$$\int_{0}^{\infty} xe^{-4x}\,dx$$

Since we have already calculated the anti-derivative, we know

$$\begin{aligned}
\int_{0}^{\infty} xe^{-4x}\,dx &= \lim_{R\to\infty} -\frac{1}{4}e^{-4x}\left(x+\frac{1}{4}\right)\Big|_{0}^{R} \\
&= \lim_{R\to\infty} -\frac{1}{4}e^{-4R}\left(R+\frac{1}{4}\right) + \frac{1}{16}
\end{aligned}$$

This illustrates a situation that often occurs when we have to do integration by parts in an improper integral. The limit above is an indeterminant form, which means we use L'Hôpital's Rule,

$$\begin{aligned}
\lim_{R\to\infty} -\frac{1}{4}e^{-4R}\left(R+\frac{1}{4}\right) &= -\frac{1}{4}\lim_{R\to\infty}\frac{R+\frac{1}{4}}{e^{4R}} \\
&= -\frac{1}{4}\lim_{R\to\infty}\frac{1}{4e^{4R}} \\
&= 0.
\end{aligned}$$

So, we can conclude

$$\int_{0}^{\infty} xe^{-4x}\,dx = \frac{1}{16}.$$

□

Problem 3. *6.6 #33.*

$$\int_{0}^{\infty} \cos x\,dx$$

Solution. This integral is improper because of the infinity in the limits of integration. We evaluate with a limit.

$$\begin{aligned}\int_0^\infty \cos x \, dx &= \lim_{R\to\infty} \int_0^R \cos x \, dx \\ &= \lim_{R\to\infty} \sin x \Big|_0^R \\ &= \lim_{R\to\infty} \sin R\end{aligned}$$

Now, sine oscillates between -1 and 1. This oscillation causes the limit to not exist since sine never settles on a value as R gets large. This is why we say this integral "diverges by oscillation."

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