Power Series Homework 1

Section 8.6

October 15, 2008

Problem 1. *Determine the interval of convergence and the function to which the power series* $\sum_{k=0}^{\infty}(3x+1)^k$ *converges.*

Solution. To determine which function this series converges to, we notice that it is a geometric series with $a = 1$ and $r = 3x + 1$, so

$$\sum_{k=0}^{\infty}(3x+1)^k = \frac{1}{1-(3x+1)} = \frac{-1}{3x}$$

Then, using the fact that the geometric series converges for $|r| < 1$, we have that the series converges when $|3x + 1| < 1$ or $-1 < 3x + 1 < 1$, $-2 < 3x < 0$, so the series converges when $-\frac{2}{3} < x < 0$.

We could also do the ratio test here,

$$\lim_{k\to\infty}\left|\frac{(3x+1)^{k+1}}{(3x+1)^k}\right| = \lim_{k\to\infty}|3x+1| < 1$$

This will again give $-\frac{2}{3} < x < 0$, but now we have to check the endpoints. If $x = 0$, the series is $\sum 1^k$ which diverges. If $x = -\frac{2}{3}$, the series is $\sum(-1)^k$ which diverges. So the interval of convergence is $-\frac{2}{3} < x < 0$.

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Problem 2. *Determine the interval and radius of convergence of*
$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k4^k}(x+2)^k$

Solution. This series is **NOT** a geometric series, so we have to use the Ratio Test.

$$\lim_{k\to\infty}\left|\frac{\frac{(-1)^{k+2}}{(k+1)4^{k+1}}(x+2)^{k+1}}{\frac{(-1)^{k+1}}{k4^k}(x+2)^k}\right| = \lim_{k\to\infty}\left|\frac{x+2}{4}\frac{k}{k+1}\right|$$

$$= \left|\frac{x+2}{4}\right| < 1$$

So, $|x+2| < 4$ or $-4 < x+2 < 4$, $-6 < x < 2$. Now, we have to check the endpoints. If $x = 2$, the series is $\sum \frac{(-1)^{k+1}}{k}$ which converges. If $x = 6$, the series is $\sum \frac{-1}{k}$ which diverges. So the interval of convergence is $(-6, 2]$ and the radius of convergence is 4.

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Problem 3. *Find a power series representation of $f(x) = 2\ln|1-x|$.*

Solution. We have a nice power series for $\frac{1}{1-x}$ and $\int \frac{1}{1-x} dx = -\ln|1-x| + C$. So, $-2\int \frac{1}{1-x} dx = 2\ln|1-x| + C$.

$$\begin{aligned} 2\ln|1-x| + C &= -2\int \frac{1}{1-x} dx = -2\int \sum_{n=0}^{\infty} x^n dx \\ &= -2\sum_{n=0}^{\infty} \int x^n dx \\ &= -2\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \end{aligned}$$

We can solve for C by setting $x = 0$ on both sides. Since $\ln 1 = 0$ and $\sum 0 = 0$, we can conclude $C = 0$. Thus,

$$2\ln|1-x| = -2\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

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