Improper Integrals Homework 2

Section 6.6

February 4, 2008

Problem 1. *Use a comparison to determine whether the integral converges or diverges.*

$$\int_1^\infty e^{-x^3} dx$$

Solution. On the interval $[1, \infty)$, we have that $0 \leq e^{-x^3} \leq e^{-x}$. Now, $\int_1^\infty e^{-x} dx$ converges (to e^{-1} in fact.) So by the Comparison Test, we have that the given integral will converge.

□

Problem 2. *If* $\lim_{x\to\infty} f(x) = 1$ *then* $\int_0^\infty f(x)dx$ *diverges.*

Solution. This is true. $\lim_{x\to\infty} f(x) = 1$ means that for all $\epsilon > 0$ there is some finite $N > 0$ such that $|F(x) - 1| < \epsilon$ whenever $x > N$. Taking for instance $\epsilon = \frac{1}{2}$, we have

$$\int_N^\infty f(x)dx > \int_N^\infty \frac{1}{2} dx = \infty$$

So this integral must diverge.

□

Problem 3. *If* $\lim_{x\to\infty} f(x) = 0$ *then* $\int_0^\infty f(x)dx$ *converges.*

Solution. This is false! Consider $f(x) = \frac{1}{x}$, $\lim_{x\to\infty} f(x) = 0$, but the integral diverges by the p-test. (In fact both p-tests since $\frac{1}{x}$ will not integrate nicely at infinity or at zero.)

□

Problem 4. $\lim_{x\to 0} f(x) = \infty$ *then* $\int_0^1 f(x)dx$ *diverges.*

Solution. This is false! Consider $f(x) = \frac{1}{x^{1/2}}$, $\lim_{x \to 0} f(x) = \infty$ but in fact, $\int_0^1 f(x)dx = 2$ and is convergent.

□

Problem 5. *If $f(-x) = -f(x)$ for all x, then $\int_{-\infty}^{\infty} f(x)dx = 0$.*

Solution. This is false! Recall that for such an integral you need to take two limits, so

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R \to \infty} \lim_{S \to -\infty} \int_S^R f(x)dx$$

with $f(x) = x$ for example, this limit will not exist. It does not exist since if R goes to infinity much faster than S goes to negative infinity, the integral should go to ∞. Yet if S goes faster, the integral should go to $-\infty$, and if they go at the same speed, the integral should go to 0. Thus, the limit will not exist and this integral diverges.

□