Sequences Homework 1

Section 8.1

February 13, 2008

Problem 1. *Conceptual Questions.*
a) Can an unbounded sequence converge?
b) Are all convergent sequence bounded?
c) What does it mean for a sequence to be **monotonic***?*
d) Are all convergent sequences monotonic?
e) Can a bounded monotonic sequence diverge?

Solution. a) The answer here is technically yes. Consider the sequence $a_n = \dfrac{1}{n-2}$, this will converge to 0 as $n \to \infty$ yet a_2 is unbounded. However, it will be "eventually bounded." Since it converges, after some number of terms in the sequence it will be close enough to the limit, and will hence be bounded. That is for and $\epsilon > 0$, there exists a constant N such that for $n \geq N$ $|a_n - L| < \epsilon$.
b) This is the roughly same question as part a) really. Technically no, but eventually all convergent sequences are bounded.
c) A sequence is monotonic if it is either increasing or decreasing.
d) No, consider $a_n = (-1)^n \frac{1}{n}$, $\lim_{n\to\infty} a_n = 0$ but a_n oscillates and cannot be monotone.
e) No, we have a theorem that tells us that all bounded monotone sequences converge.

□

Problem 2. *Does* $a_n = \dfrac{3^n}{e^n + 1}$ *converge or diverge?*

Solution.

$$\begin{aligned} \frac{3^n}{e^n+1} &= \frac{3^n}{e^n+1} \cdot \frac{3^{-n}}{3^{-n}} \\ &= \frac{1}{(\frac{e}{3})^n + e^{-n}} \end{aligned}$$

Taking the limit, we have

$$\lim_{n\to\infty} \frac{3^n}{e^n+1} = \lim_{n\to\infty} \frac{1}{(\frac{e}{3})^n + e^{-n}}$$

which diverges since the denominator of this fraction goes to zero.

□

Problem 3. *Does* $a_n = \sqrt{n^2+n} - n$ *converge or diverge?*

Problem 4. *We could use L'Hôpital on the function* $f(x) = \sqrt{x^2+x} - x$ *to see that the limit will go to* $\frac{1}{2}$. *Alternatively, we could multiply by a convenient value of 1.*

$$\begin{aligned}\sqrt{n^2+n} - n &= (\sqrt{n^2+n} - n)\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} \\ &= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} \\ &= \frac{n}{\sqrt{n^2+n}+n} \\ &= \frac{n}{\sqrt{n^2+n}+n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\ &= \frac{1}{\sqrt{1+\frac{1}{n}}+1}\end{aligned}$$

Taking the limit we see

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} \\ &= \frac{1}{2}\end{aligned}$$

Problem 5. *(From the worksheet.) Suppose that* $a_1 = 1$ *and* $a_{n+1} = \dfrac{1}{2}(a_n + \dfrac{4}{a_n})$.
a) Show that this sequence converges to 2.

Solution. Let L be the limit $\lim\limits_{n\to\infty} a_n = L$. Then $\lim\limits_{n\to\infty} a_{n+1} = L$. So,

$$\begin{aligned}\lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \frac{1}{2}(a_n + \frac{4}{a_n}) \\ L &= \frac{1}{2}(L + \frac{4}{L}) \\ \frac{1}{2}L &= \frac{2}{L}\end{aligned}$$

We can see that $L = \pm 2$ are the only possibilities for the limit.

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