SOLUTIONS TO REVIEW FOR EXAM 1

Notes on the Test: I will mostly be grading your work, not your final answers. This means that I'm fairly generous about small algebra mistakes, but having the correct answer at the end is worth nothing, *a priori.* Here's an example problem, with several graded solutions:
Problem Is $\int_0^1 x^2 dx \leq M_{50}$, where M_{50} denotes a midpoint approximation to this integral? Explain why.
Answer 1. No. (0/10 points for the correct answer)
Answer 2. Yes because $y = x^2$ is concave down on this interval, and so by a theorem M_n is an overestimate. (6/10 points–right idea, but wrong concavity)
Answer 3. No. $\frac{d^2}{dx^2}x^2 = 2 > 0$, so f is concave up. By a theorem M_n is an underestimate where concave up. (10/10 points)
Note that the "correct" answer by itself is worth nothing, and that an "incorrect" answer is worth more than half the points. (Possibly even worth all the points if the circumstance is right.) If I ask you to explain something, and a theorem applies, you don't need to explain why the theorem is true but you do need to explain *why the theorem applies.* In this case, that means computing the second derivative and finding the concavity of the function on the interval $[0, 1]$.

(1) Let $f(x) = x^3$, $I = \int_1^4 f(x)dx$.
  (a) Compute L_3, R_3, T_3 and M_3.
  Solution: Here $\Delta x = 1$, $x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4$. Thus:

$$L_3 = \sum_{i=0}^{3-1} f(x_i))\Delta x = 1^3 \cdot 1 + 2^3 \cdot 1 + 3^3 \cdot 1 = 1 + 8 + 27 = 36$$

$$R_3 = \sum_{i=1}^{3} f(x_i)\Delta x = 2^3 \cdot 1 + 3^3 \cdot 1 + 4^3 \cdot 1 = 8 + 27 + 64 = 99$$

$$T_3 = (L_3 + R_3)/2 = \frac{135}{2} = 67.5$$

$$M_3 = \sum_{i=0}^{3-1} f((x_i + x_{i+1})/2)\Delta x = (1.5)^3 \cdot 1 + (2.5)^3 \cdot 1 + (3.5)^3 \cdot 1 = 61.875$$

  (b) If possible, write an inequality bounding I between L_{25} and R_{50} and an inequality bounding I between T_{50} and M_{30}. Explain why these inequalities are true.
  Solution: $f'(x) = 3x^2 > 0$ on $[1, 4]$, so f is increasing. By a theorem covered in class, L_{25} is an underestimate, and R_{50} is an overestimate. Thus $L_{25} \leq I \leq R_{50}$. $f''(x) = 6x > 0$ on $[1, 4]$, so f'' is concave up on $[1, 4]$. Thus by a theorem proved in class, the trapezoidal approximations are overestimates, and the midpoint approximations are underestimates. Thus $M_{30} \leq I \leq T_{50}$.

(2) If f is a function for which $10^{-10} < f''(x) < 10^{-6}$ on $[1, 3]$, is M_{50} a good approximation to $\int_1^3 f(x)dx$? Explain why.
Solution: By a theorem proved in class, the (worst case) approximation error of the midpoint approximation is inversely proportional to the square of the number of intervals, and directly proportional to the length of the interval and the bound on the absolute value of the second derivative. Thus since $|f''| < 10^{-6}$, the interval is short, and we have a fairly large number of subdivisions, the approximation error is small. Therefore, M_{50} is a good approximation. (The key point here is the bound on the second derivative. Less wordy answers would also result in full credit.)

(3) Find the arc length of the following curves on the following intervals:

(a) $y = x^{3/2}$ on $[1, 2]$.
Solution: $y' = (3/2)x^{3/2-1} = (3/2)x^{1/2}$. Thus the arc length is:

$$\int_1^2 \sqrt{1+(y')^2}dx = \int_1^2 \sqrt{1+(9/4)x}dx$$

Letting $u = 1 + (9/4)x$, so that $du = (9/4)dx$, we see that the above integral is equal to:

$$\frac{4}{9}\int_{1+9/4}^{2+9/4} \sqrt{u}du = \frac{4}{9}\left[\frac{2}{3}x^{3/2}\right]_{13/4}^{17/4}$$

You do not need to finish evaluating the integral to get full credit.

(b) $y = \sqrt{1-x^2}$ on $[-1, 1]$.
Solution: $y' = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$. Thus the arc length is given by:

$$\int_{-1}^1 \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}dx = \int_{-1}^1 \sqrt{1+\frac{x^2}{1-x^2}}dx.$$

Finding a common denominator and simplifying, we get:

$$\int_{-1}^1 \sqrt{\frac{(1-x^2)+x^2}{1-x^2}}dx = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}}dx = [\arcsin x]_{-1}^1 = \frac{\pi}{2} - \frac{-\pi}{2} = \pi$$

Again, you do not need to finish evaluating the integral to get full credit.

(c) $y = 3x + 2$ on $[1, 3]$.
Solution: $y' = 3$, so the arc length is given by

$$\int_1^3 \sqrt{1+3^2}dx = \left[\sqrt{10}x\right]_1^3 = 2\sqrt{10}$$

(4) Find the volume of the following solids:

(a) The base of S in the xy-plane is the region bounded by $y = x^2/2$ and the line $y = 2$, and cross-sections of S perpendicular to the x-axis are semicircles.
Solution: The line $y = 2$ intersects $y = x^2/2$ at the points $(\pm 2, 2)$. The area function at the cross section $x = x_0$ is given by $A(x_0) = \pi \cdot ((2 - x_0^2/2)/2)^2 = \frac{\pi}{4}(2 - x_0^2/2)^2$. Thus the volume is given by:

$$\int_{-2}^2 A(x)dx = \frac{\pi}{4}\int_{-2}^2 (4 - 2x^2 + x^4/4)dx = \frac{\pi}{4}\left[2x - \frac{2}{3}x^3 + \frac{1}{20}x^5\right]_{-2}^2.$$

(b) The region bounded by $y = x$, $x = 3$ and the x-axis, revolved around the y-axis.
Solution 1: We can use washers. Here the outer radius is 3, and the inner radius is y, and the bounds are $y = 0$ to $y = 3$. Thus the volume is given by:

$$V = \int_0^3 \pi(3^2 - y^2)dy = \pi\left[9y - \frac{1}{3}y^3\right]_0^3 = \pi\left(27 - \frac{27}{3}\right) = 18\pi.$$

Solution 2: We can use cylindrical shells. Here the radius at x is just x, and the height is just x, with bounds $x = 0$ to $x = 3$. Then the volume is given by:

$$V = \int_0^3 2\pi(x)(x)dx = 2\pi\left[\frac{1}{3}x^3\right]_0^3 = \frac{2\pi \cdot 27}{3} = 18\pi$$

Solution 3: We can remember our high school geometry, and recognize that this solid is just a cylinder of height and radius 3 with a cone of height and radius 3 cut out of it. Then

$$V = 3 \cdot 3^2 \cdot \pi - \left(\frac{1}{3}\right)3 \cdot 3^2 \cdot \pi = 18\pi.$$

To receive full credit, this last method needs a bit of explanation about what you are doing.

(c) Show that the volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.
Solution: We want to build a cone of height h and radius r by rotating a region around an axis. There are a few ways to achieve this; probably the simplest one is by rotating the region bounded by the the x and y axes, and a line with x intercept h and y intercept r around the x-axis. Since this line contains the points $(0, r)$ and $(h, 0)$, it has slope $-r/h$, and so it has equation $y = \frac{-r}{h}x + r$. Then we can use the method of disks: the radius at x is given by $\frac{-r}{h}x + r$ and we are integrating with respect to x from $x = 0$ to $x = h$. Thus:

$$
\begin{aligned}
V &= \int_0^h \pi \left(\frac{-r}{h}x + r\right)^2 dx \\
&= \pi \int_0^h \left(\frac{r^2}{h^2}x^2 - \frac{2r^2}{h}x + r^2\right) dx \\
&= \pi \left[\frac{r^2x^3}{3h^2} - \frac{2r^2x^2}{2h} + r^2x\right]_0^h \\
&= \pi \left(\frac{r^2h^3}{3h^2} - \frac{r^2h^2}{h} + r^2h\right) \\
&= \pi(r^2h/3 - r^2h + r^2h) = \pi(r^2h/3) = \frac{1}{3}\pi r^2 h.
\end{aligned}
$$

(5) Find the area of region bounded by the curves $y = 4 - x^2$ and $y = x^2 - 4$.
Solution: These curves intersect that $(\pm 2, 0)$, and the former is "on top". Thus the area is given by:

$$\int_{-2}^{2} (\text{“top”} - \text{“bottom”})dx = \int_{-2}^{2} (4 - x^2 - (x^2 - 4))dx = \int_{-2}^{2} (8 - 2x^2)dx = \left[8x - \frac{2x^3}{3}\right]_{-2}^{2} = \frac{64}{3}.$$

(6) Solve the following differential equations exactly:

(a) $y' = \frac{\sqrt{y}}{1+t^2}$.
Solution:

$$
\begin{aligned}
\frac{dy}{dt} &= \frac{\sqrt{y}}{1+t^2} \\
y^{-1/2}dy &= \frac{dt}{1+t^2} \\
\int y^{-1/2}dy &= \int \frac{dt}{1+t^2} \\
2\sqrt{y} &= \arctan t + C \\
y &= \frac{1}{4}(\arctan t + C)^2
\end{aligned}
$$

(b) $y' = ty^3$.
Solution:

$$
\begin{aligned}
\frac{dy}{dt} &= ty^3 \\
y^{-3}dy &= tdt \\
\int y^{-3}dy &= \int tdt \\
\frac{-1}{2}y^-2 &= t^2/2 + C \\
y &= \pm(-2(t^2/2 + C))^{-1/2}
\end{aligned}
$$

Note that this solution is defined only where $-2(t^2/2 + C) > 0$.

(c) $y' = (\sin t)(1-y)$.
Solution:

$$\begin{aligned}
\int \frac{dy}{1-y} &= \int \sin t dt \\
-\ln|1-y| &= -\cos t + C \\
|1-y| &= e^{\cos t - C} \\
1-y &= \pm e^{\cos t - C} \\
y &= 1 \pm e^{\cos t - C}
\end{aligned}$$

(7) Apply Euler's method with 3 subdivisions to find $y(4)$ in the following IVP: $y' = y^2$, $y(1) = 2$.
Solution: Note that $\Delta t = 1$.

$$\begin{aligned}
y_0 &= 2 \\
y_1 &= y_0 + (y_0)^2 \cdot \Delta t = 2 + 2^2 \cdot 1 = 6 \\
y_2 &= y_1 + (y_1)^2 \cdot \Delta t = 6 + 6^2 \cdot 1 = 42 \\
y_3 &= y_2 + (y_2)^2 \cdot \Delta t = 42 + 42^2 \cdot 1 = 1806
\end{aligned}$$

Note that this approximation is horrible: the correct answer is only $\frac{-2}{5}$! The reason is that there is an asymptote (in the correct solution) at $x = 3/2$, and Euler's method cannot say very much about discontinuous functions. Also, the second derivative near this asymptote is large, and Euler's method does not do very well with functions where the derivative is changing rapidly. If a problem like this appears on the exam, it will be better chosen. Engineers take note: just because your formulas tell you something doesn't mean it corresponds to reality. Learn a sense of the scale of the right answer.

(8) Find the following integrals:

(a) $\int \sin x \cos x \ln \sin x dx$
Solution: Let $y = \sin x$, so $dy = \cos x dx$. Then:

$$\begin{aligned}
\int \sin x \cos x \ln \sin x dx &= \int y \ln y dy \\
&= \frac{y^2}{2} \ln y - \int \frac{y^2}{2} \cdot \frac{1}{y} dy \qquad (u = \ln y,\ dv = y dy) \\
&= \frac{y^2}{2} \ln y - y^2/4 + C \\
&= \frac{\sin^2 x}{2} \ln \sin x - \sin^2 x/4 + C
\end{aligned}$$

(b) $\int x^5 e^{-x^3} dx$.
Solution: Let $y = x^3$, so that $dy = 3x^2$. Then:

$$\begin{aligned}
\int x^5 e^{-x^3} dx &= \frac{1}{3} \int y e^{-y} dy \\
&= \frac{1}{3} \left(-y e^{-y} - \int (-e^{-y}) dy \right) \qquad (u = y,\ dv = e^{-y} dy) \\
&= \frac{1}{3} \left(-y e^{-y} - e^{-y} \right) + C \\
&= \frac{1}{3} \left(-x^3 e^{-x^3} - e^{-x^3} \right) + C
\end{aligned}$$

(c) $\int_1^e x \ln x dx$. We choose $u = \ln x$, $dv = xdx$, so that $du = dx/x$, and $v = x^2/2$. Then:

$$\begin{aligned} \int_1^e x \ln x &= \left[\frac{x^2 \ln x}{2}\right]_1^e - \int_1^e \frac{x^2 dx}{2x} \\ &= e^2/2 - \left[\frac{x^2}{4}\right]_1^e \\ &= e^2/2 - (e^2/4 - 1/4) = e^2/4 + 1/4. \end{aligned}$$

(d) Show that $\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$, and use this formula to find $\int x^2 e^x dx$.
Solution: Let $u = x^n$, $dv = e^x dx$. Then $du = nx^{n-1} dx$, $v = e^x$. Thus:

$$x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx.$$

Applying this formula twice yields:

$$\begin{aligned} \int x^2 e^x dx &= x^2 e^x - 2(\int x e^x dx) \\ &= x^2 e^x - 2(x e^x - \int e^x dx) \\ &= x^2 e^x - 2(x e^x - e^x) + C \end{aligned}$$