REVIEW FOR EXAM 2

1. Notes about Exam:

The basic format of the exam will be similar to the first exam. You may not use calculators, notes, etc. Any trigonometric identies you need will be included on the exam (though you might need to alter them slightly in order to make them useful to your situation). This topics covered include the following:

- Integration of rational functions (partial fractions, trig substitutions)
- Integration of trigonometric functions
- Definition and computation of Taylor polynomials
- Taylor's theorem and applications. (Note that this is different from the definition of Taylor polynomials, something a few people seem confused about.) You must be able to understand what sort of bound is required and how to find it in simple circumstances (eg. sine and cosine, increasing functions, etc).
- Improper integrals, definition of convergence for improper integrals, comparison test for integrals.
- Seqences. Properties of sequences (monotonicity, boundedness), definition of convergence of a seqence.
- Series. Definition of convergence for a series. Geometric series. Integral test, comparison test, ratio test.

Any of these topics is fair game on the exam, though some of them may not actually be on the exam. A few questions on the exam will be either verbatim off this review or slightly altered, so it would be highly beneficial to go through this review in some detail.

Some solutions are not as detailed as I would like yours to be on the exam. For example, when I say "such and such integral converges" you would have have to show that the integral actually converges either with the comparison test or by directly computing it. You may assume the following series convergence/divergence on the exam without proof, *unless I ask you to show that one of these series converges or diverges*:

- $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$, diverges if $p \leq 1$.
- $\sum_{k=0}^{\infty} a \cdot r^k$ converges if $|r| < 1$, diverges if $|r| \geq 1$.

2. Review Problems

(1) (Integration of rational functions) Evaluate the following:

(a) $\int \frac{dx}{x^2 + 2x + 3}$

Solution: The quadratic formula tells us that this does not factor into linear factors (since $2^2 - 4 \cdot 3 \cdot 1 < 0$, the denominator has no real roots). We can however, complete the square: $x^2 + 2x + 3 = (x + 1)^2 + 2$. Now finish the computation with the substitution $u = (x + 1)/\sqrt{2}$. (This should be a very routine calculation by now).

(b) $\int \frac{2dx}{x^2 - x - 6}$

Solution: The denominator factors as $(x - 3)(x + 2)$. We want to solve:

$$\frac{2}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} = \frac{(A+B)x + (2A - 3B)}{(x-3)(x+2)}$$

In other words, we have the equations $A + B = 0$, and $2A - 3B = 2$. By the first equation, we may substitute $B = -A$ into the second equation, obtaining $5A = 2$, so $A = 2/5$, and

$B = -2/5$. Thus:

$$\int \frac{2dx}{x^2 - x - 6} = \frac{2}{5}\left(\int \frac{dx}{x-3} - \int \frac{dx}{x+2}\right) = \frac{2}{5}\left(\ln|x-3| - \ln|x+2|\right)$$

(c) $\int \frac{2x^2 - x + 1}{(x-1)(x^2+1)} dx$

Solution: First note taht $x^2 + 1$ does not split into factors, so we want to solve

$$\frac{2x^2 - x + 1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2+1} = \frac{(A+B)x^2 + (C-B)x + (A-C)}{(x-1)(x^2+1)}$$

This gives three equations: $A + B = 2$, $C - B = -1$, $A - C = 1$. The second equation gives that $B = C + 1$, and the third gives that $A = C + 1$, so $A = B$. Since $A + B = 2$, $A = B = 1$, and $C = 0$. Thus:

$$\int \frac{2x^2 - x + 1}{(x-1)(x^2+1)} dx = \int \frac{dx}{x-1} + \int \frac{xdx}{x^2+1} = \ln|x-1| + \frac{1}{2}\ln|x^2+1|$$

(The second integral is found via the substitution $u = x^2 + 1$).

(2) (Trig. Antiderivatives) Evaluate the following:

(a) $\int \tan^3 x dx$.

Solution: Recall that $1 + \tan^2 x = \sec^2 x$, so $\tan^2 x = \sec^2 x - 1$. Thus:

$$\begin{aligned} \int \tan^3 x dx &= \int \tan^2 x \tan x dx \\ &= \int (\sec^2 x - 1)\tan x dx \\ &= \int \tan x \sec^2 x dx - \int \tan x dx \\ &= \frac{1}{2}\tan^2 x + \ln|\cos x| \end{aligned}$$

The first integral can be easily evaluated by substituting $u = \tan x$, so $du = \sec^2 x dx$. The second can be evaluated by writing $\tan x = \sin x / \cos x$, and using the substitution $v = \cos x$.

(b) $\int_{-\pi}^{\pi} \sin^2(3x) dx$

Solution: Here we want to use a double angle identity, which gives us: $\sin^2 3x = \frac{1}{2} - \frac{1}{2}\cos(6x)$. Thus:

$$\int_{-\pi}^{\pi} \sin^2(3x) dx = \left[\frac{x}{2} - \frac{1}{12}\sin(6x)\right]_{-\pi}^{\pi} = \pi$$

(c) $\int \cos^3 x dx$

Solution: Since $\sin^2 x + \cos^2 x = 1$, $\cos^3 x = (1 - \sin^2 x)\cos x$. Thus:

$$\begin{aligned} \int \cos^3 x dx &= \int (1 - \sin^2 x)\cos x dx \\ &= \int (1 - u^2) du \qquad (u = \sin x) \\ &= u - \frac{u^3}{3} = \sin x - \frac{\sin^3 x}{3} \end{aligned}$$

(3) (Taylor Polynomials & Taylor's theorem)

(a) Find the fifth Taylor polynomial of $f(x) = \sin x$ centered at $\frac{5\pi}{4}$.

Solution: $f'(x) = \cos x$, $f^{(2)}(x) = -\sin x$, $f^{(3)}(x) = -\cos x$, $f^{(4)}(x) = \sin x$, and $f^{(5)}(x) = \cos x$. Note that $\cos \frac{5\pi}{4} = \frac{-\sqrt{2}}{2}$, and $\sin \frac{5\pi}{4} = \frac{-\sqrt{2}}{2}$. Thus:

$$P_5(x) = \frac{-\sqrt{2}}{2}\left(1 + \left(x - \frac{5\pi}{4}\right) - \frac{\left(x - \frac{5\pi}{4}\right)^2}{2!} - \frac{\left(x - \frac{5\pi}{4}\right)^3}{3!} + \frac{\left(x - \frac{5\pi}{4}\right)^4}{4!} + \frac{\left(x - \frac{5\pi}{4}\right)^5}{5!}\right)$$

(b) Use Taylor's theorem to find an interval $[\frac{5\pi}{4} - c, \frac{5\pi}{4} + c]$ such that the approximation error of the polynomial you got in part (a) is less than $1/100$ for all x in this interval.
Solution: We know that the derivatives of sin repeat, and all of them are $\pm \sin x$ or $\pm \cos x$. Thus $|f^{(k)}(x)| \leq 1$ for all x. In particular, $|f^{(6)}(x)| \leq 1$ for all x, so in particular for all x in $I = [\frac{5\pi}{4} - c, \frac{5\pi}{4} + c]$. (I.E. In the statement of Taylor's theorem, we can take $K = 1$). Further note that if x is in the interval I, then $|x - \frac{5\pi}{4}| \leq c$, so that, by Taylor's theorem, we have:

$$|P_5(x) - \sin x| \leq \frac{1}{(5+1)!} \cdot |x - 5\pi/4|^{5+1} \leq \frac{1}{6!} c^6.$$

So if we find a c such that $\frac{c^6}{6!} \leq \frac{1}{100}$, we are done. Multiplying both sides of the inequality by $6!$, and taking 6^{th} roots, we get that $c \leq \left(\frac{6!}{100}\right)^{1/6} \approx 1.39$. So in particular $c = 1.39$ works. (Other values for c are acceptable as long as they are supported by a similar calculation. For example $c = 1$ or $c = 1.2$ would also work fine.)

(c) Find the third Maclaurin polynomial of $f(x) = e^{2x}$.
Solution: $P_3(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3$. (You should get that $f^{(k)}(x) = 2^k e^{2x}$.)

(d) Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ be a polynomial of degree n. Show that the n^{th} Taylor polynomial (centered at x_0) is equal to $f(x)$. (Hint: Use Taylor's theorem.)
Solution: The k^{th} derivative of f is a polynomial of degree $n-k$ (since each derivative reduces the degree by 1. Thus the n^{th} derivative is a degree zero polynomial (i.e. a constant), and so the $(n+1)$ derivative is zero. Thus in particular the $(n+1)$ derivative is bounded by $K = 0$, so by Taylor's theorem:

$$|f(x) - P_n(x)| \leq \frac{0}{(n+1)!} \cdot |x - x_0|^{n+1} = 0$$

So $|f(x) - P_n(x)| = 0$, i.e. $f(x) = P_n(x)$.

(4) (Fourier Polynomials)

(a) Suppose that $f(x)$ is an even function, and $\int_0^\pi f(x)dx = 1$, $\int_0^\pi f(x)\cos x dx = 3$, and $\int_{-\pi}^0 f(x)\cos(2x)dx = 5$. Find the second Fourier polynomial of f.
Solution: Since f is an even function, all coefficients of $\sin(mx)$ are zero. Furthermore,

$$\int_{-\pi}^{\pi} f(x)dx = 2\int_0^\pi f(x)dx = 2$$

Similarly, $\int_{-\pi}^{\pi} f(x)\cos x dx = 6$, and $\int_{-\pi}^{\pi} f(x)\cos(2x)dx = 10$. Thus the second fourier polynomial is $\frac{1}{\pi} \cdot (1 + 6\cos x + 10\cos(2x))$.

(b) (i) Show that $\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = 0$, where $n \neq m$ positive integers. (Hint: Use the identity $2\cos u \cdot \cos v = \cos(u+v) - \cos(u-v)$).
Solution:

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx &= \frac{1}{2}\int_{-\pi}^{\pi} (\cos((m+n)x) - \cos((m-n)x))\, dx \\
&= \frac{1}{2}\left[\frac{1}{m+n}\sin((m+n)x) - \frac{1}{m-n}\sin((m-n)x)\right]_{-\pi}^{\pi} \\
&= 0 + 0 \qquad\qquad (\text{if } m \pm n \neq 0)
\end{aligned}$$

(ii) Evaluate $\int_{-\pi}^{\pi} \cos^2(mx)dx$.
Solution:

$$\int_{-\pi}^{\pi} \cos^2(mx)dx = \frac{1}{2}\left(\int_{-\pi}^{\pi}(1 + \cos(2mx))dx\right) = \pi$$

(iii) Find the fourth Fourier polynomial of $2 + \cos x + 4\cos(2x) + \cos(5x)$.
Solution: Using parts (i) and (ii), we see that the answer is $2 + \cos x + 4\cos(2x)$.

(c) Find the fourth Fourier polynomial of $f(x) = x^2 + 1$.

(d) This is an even function, so we only need to worry about the cos coefficients. The constant coefficient is

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x^2+1)dx = \frac{1}{2\pi}\left[\frac{x^3}{3}+x\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}+1$$

Also for $k > 0$:

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+1)\cos(kx)dx \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} x^2\cos(kx)dx + \int_{-\pi}^{\pi}\cos(kx)dx\right) \\ &= \frac{1}{\pi}\left(\left[\frac{x^2\sin(kx)}{k}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{2x\sin(kx)}{k}dx\right) \\ &= \frac{1}{\pi}\left(0 - \left(\left[\frac{-2x\cos(kx)}{k^2}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{-2\cos(kx)}{k^2}dx\right)\right) \\ &= \frac{1}{\pi}\cdot\frac{4\pi\cos(k\pi)}{k^2} = \frac{(-1)^k\cdot 4}{k^2} \end{aligned}$$

Thus the fourth Fourier polynomial of $f(x)$ is:

$$q_4(x) = \left(\frac{\pi^2}{3}+1\right) - 4\cos x + \cos(2x) - \frac{4}{9}\cos(3x) + \frac{1}{4}\cos(4x)$$

(5) (Improper integrals)

(a) Let f be a continuous function. Write down the definition of what it means that $\int_0^\infty f(x)dx = I$.

Solution: $\lim_{t\to\infty}\int_0^t f(x)dx = I$

(b) Evaluate $\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$.

Solution:

$$\begin{aligned} \int_{-\infty}^{\infty}\frac{dx}{1+x^2} &= \lim_{t\to\infty}[\arctan x]_0^t + \lim_{t\to-\infty}[\arctan x]_t^0 \\ &= (\frac{\pi}{2}-0)+(0-\frac{-\pi}{2}) = \pi \end{aligned}$$

(c) Evaluate $\int_{-\infty}^{\infty} e^{-|x|}dx$.

Solution:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-|x|}dx &= \lim_{t\to\infty}\int_0^t e^{-x}dx + \lim_{t\to-\infty}\int_t^0 e^{x}dx \\ &= \lim_{t\to\infty}\left[-e^{-x}\right]_0^t + \lim_{t\to-\infty}\left[e^{x}\right]_t^0 \\ &= (0-(-1))+(1-0) = 2 \end{aligned}$$

(d) Evaluate $\int_{-1}^{\infty} x^{-2}dx$.

Solution:

$$\int_{-1}^{\infty} x^{-2}dx = \lim_{t\to0^-}\int_{-1}^{t} x^{-2}dx + \lim_{t\to0^+}\int_{t}^{1} x^{-2}dx + \lim_{t\to\infty}\int_{1}^{t} x^{-2}dx$$

Simple calculations (which you would have to do in full detail on the exam) show that the first two limits don't converge, the third does, and so the integral itself does not converge.

(e) Show that $\int_1^\infty \frac{dx}{x^4+2}$ converges.

Solution: $x^4+2 \geq x^4$, so $0 \leq 1/(x^4+2) \leq 1/x^4$. Thus $0 \leq \int_1^\infty \frac{dx}{x^4+2} \leq \int_1^\infty \frac{dx}{x^4}$. We can (and you should for the test) check that the second integral converges, so the first one must as well, by the comparison test.

(f) Show that $\int_1^\infty \frac{dx}{2x-1}$ diverges.
Solution: Note that $\int \frac{dx}{2x-1} = \frac{1}{2}\ln|2x-1| + C$ by the substitution $u = 2x-1$. Thus:

$$\int_1^\infty \frac{dx}{2x-1} = \lim_{t\to\infty}\left[\frac{1}{2}\ln|2x-1|\right]_1^t = \infty$$

Thus the integral diverges.

(6) (Sequences)

(a) Exercises 19-24, section 11.1.
"**Solution:**" Gosh, these would make *fantastic* exam questions, but if I give some answers here, then those will be the only answers I see on the exam. Since variety is the spice of life, I think I'll let you figure some examples on your own. (I will be more than happy to tell you if an example is right.)

(b) Let $a_1 = 1$, $a_{n+1} = \frac{n}{n+1}\cdot a_n$.

(i) Show that this sequence is bounded and monotone.
Solution: Multiplying a bunch of numbers in $[0,1]$ also yields a number in $[0,1]$, so $|a_n| \le 1$ for all n. Thus the seqence is bounded. It is monotone since $0 \le n/(n+1) \le 1$, so $a_{n+1} = \frac{n}{n+1}a_n \le 1\cdot a_n = a_n$. Thus the sequence is monotone decreasing.

(ii) Use part (a) to show that it converges.
Solution: Every bounded monotone sequence converges (theorem discussed in class).

(iii) Evaluate the limit of the sequence (hint: write down the first few terms, and look for a pattern.)
Solution: Note that:

$$a_n = 1\cdot\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n-2}{n-1}\cdot\frac{n-1}{n} = \frac{1}{n}$$

Thus $\lim_{n\to\infty} a_n = 0$.

(c) Find the limit of the sequence $a_k = \int_k^\infty \frac{dx}{1+x^2}$.
Solution:

$$a_k = \int_k^\infty \frac{dx}{1+x^2} = \lim_{t\to\infty}(\arctan t - \arctan k) = \frac{\pi}{2} - \arctan k$$

Thus:

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}\left(\frac{\pi}{2} - \arctan k\right) = \frac{\pi}{2} - \frac{\pi}{2} = 0$$

(d) Find the limit of the sequence $a_k = \sqrt{k^2+1} - k$.

$$\begin{aligned}
\lim_{k\to\infty} a_k &= \lim_{k\to\infty} \sqrt{k^2+1} - k \\
&= \lim_{k\to\infty} \frac{(\sqrt{k^2+1}-k)(\sqrt{k^2+1}+k)}{\sqrt{k^2+1}+k} \\
&= \lim_{k\to\infty} \frac{k^2+1-k^2}{\sqrt{k^2+1}+k} \\
&= \lim_{k\to]infty} \frac{1}{\sqrt{k^2+1}+k} = 0 \qquad \text{(because } \sqrt{k^2+1}+k\to\infty)
\end{aligned}$$

(7) (Series)

(a) Evaluate the following series:

(i) $\sum_{k=4}^\infty 5^{-k}$.
Solution: $\sum_{k=4}^\infty 5^{-k} = 5^{-4}\sum_{k=0}^\infty 5^{-k} = 5^{-4}\cdot\frac{1}{1-1/5}$.

(ii) $\sum_{k=0}^\infty 5^k$.
Solution: ∞

(iii) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+2)(k+3)}$.

Solution: First we need to find the partial fractions decomposition of $\frac{1}{(k+2)(k+3)}$:

$$\frac{1}{(k+2)(k+3)} = \frac{A}{k+2} + \frac{B}{k+3} = \frac{(A+B)k + (3A+2B)}{(k+2)(k+3)}$$

This has solutions $\frac{1}{(k+2)(k+3)} = \frac{1}{k+2} - \frac{1}{k+3}$. Thus:

$$\begin{aligned} S_n = \sum_{k=1}^{n} \frac{1}{(k+2)(k+3)} &= \sum_{k=1}^{n} \frac{1}{k+2} - \sum_{k=1}^{n} \frac{1}{k+3} \\ &= \sum_{k=3}^{n+2} \frac{1}{k} - \sum_{k=4}^{n+3} \frac{1}{k} \\ &= \frac{1}{3} - \frac{1}{n+3} \end{aligned}$$

Thus:

$$\sum_{k=1}^{\infty} \frac{1}{(k+2)(k+3)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{3} - \frac{1}{n+3}\right) = \frac{1}{3}$$

(b) Determine whether the following series converge/diverge in the specified number of ways:

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k^2+1}$. (2 ways)

Solution: (1) Comparison test: $1/(2k^2+1) \leq 1/(2k^2)$.

$$\sum_{k=1}^{\infty} \frac{1}{2k^2+1} \leq \sum_{k=1}^{\infty} \frac{1}{2k^2} = 2\sum_{k=1}^{\infty} \frac{1}{k^2}, \text{ which converges.}$$

(2) Integral test: To evaluate the integral $\displaystyle\int_1^{\infty} \frac{dx}{2x^2+1}$, use the substitution $u = \sqrt{2}x$. (The computation of this integral is omitted, but should be included for full credit on the exam.)

(ii) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{3^k+2}$. (2 ways)

Solution: (1) Comparison test: $1/(3^k+2) \leq 1/3^k$, and $\displaystyle\sum_{k=0}^{\infty} \frac{1}{3^k}$ converges. (See previous solution for complete work.)

(2) Ratio test:

$$\lim_{k\to\infty} \frac{1/(3^{k+1}+2)}{1/(3^k+2)} = \lim_{k\to\infty} \frac{3^k+2}{3^{k+1}+2} = \frac{1}{3}$$

(iii) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$. (1 way)

Solution: Integral test: show that $\int_1^{\infty} \frac{dx}{x}$ diverges (show work for full credit on exam).

(iv) $\frac{4}{7^{10}} + \frac{4}{7^{12}} + \frac{4}{7^{14}} + \frac{4}{7^{16}} + \ldots$ (3 ways).

Solution: (1) (Evaluate). This is just a geometric series with $a = 4/7^{10}$, and $r = 7^{-2} < 1$. Thus this series converges.

(2) (Ratio test). The ratio of successive terms is $7^{-2} < 1$.

(3) (Integral test). Evaluate the integral $\int_5^{\infty} 4 \cdot 7^{-2x} dx$, and show it converges.

(v) $\displaystyle\sum_{k=1}^{\infty} \frac{\arctan k}{1+k^2}$ (1 way)

(1) (Integral test). Use the substitution $u = \arctan x$ in the following integral:

$$\int_1^{\infty} \frac{\arctan x}{1+x^2} dx = \int_{\pi/4}^{\pi/2} u du < \infty$$

So by the integral test the series converges.

(2) (Comparison test). For $k > 0$, $0 \leq \arctan k \leq \frac{\pi}{2}$, and $1/(k^2+1) \leq 1/k^2$, so

$$0 \leq \sum_{k=1}^{\infty} \frac{\arctan k}{k^2+1} \leq \sum_{k=1}^{\infty} \frac{\pi/2}{k^2} = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{1}{k^2}$$

We know the last series converges, hence so does the first.