REVIEW FOR EXAM 2

1. Notes about Exam:

The basic format of the exam will be similar to the first exam. You may not use calculators, notes, etc. Any trigonometric identies you need will be included on the exam (though you might need to alter them slightly in order to make them useful to your situation). This topics covered include the following:

- Integration of rational functions (partial fractions, trig substitutions)
- Integration of trigonometric functions
- Definition and computation of Taylor polynomials
- Taylor's theorem and applications. (Note that this is different from the definition of Taylor polynomials, something a few people seem confused about.) You must be able to understand what sort of bound is required and how to find it in simple circumstances (eg. sine and cosine, increasing functions, etc).
- Improper integrals, definition of convergence for improper integrals, comparison test for integrals.
- Seqences. Properties of sequences (monotonicity, boundedness), definition of convergence of a seqence.
- Series. Definition of convergence for a series. Geometric series. Integral test, comparison test, ratio test.

Any of these topics is fair game on the exam, though some of them may not actually be on the exam. A few questions on the exam will be either verbatim off this review or slightly altered, so it would be highly beneficial to go through this review in some detail.

Some solutions are not as detailed as I would like yours to be on the exam. For example, when I say "such and such integral converges" you would have have to show that the integral actually converges either with the comparison test or by directly computing it. You may assume the following series convergence/divergence on the exam without proof, *unless I ask you to show that one of these series converges or diverges*:

- $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$, diverges if $p \leq 1$.
- $\displaystyle\sum_{k=0}^{\infty} a \cdot r^k$ converges if $|r| < 1$, diverges if $|r| \geq 1$.

2. Review Problems

(1) (Integration of rational functions) Evaluate the following:
   (a) $\displaystyle\int \frac{dx}{x^2+2x+3}$
   (b) $\displaystyle\int \frac{2dx}{x^2-x-6}$
   (c) $\displaystyle\int \frac{2x^2-x+1}{(x-1)(x^2+1)}dx$

(2) (Trig. Antiderivatives) Evaluate the following:
   (a) $\int \tan^3 x dx$.
   (b) $\int_{-\pi}^{\pi} \sin^2(3x)dx$
   (c) $\int \cos^3 x dx$

(3) (Taylor Polynomials & Taylor's theorem)
   (a) Find the fifth Taylor polynomial of $f(x) = \sin x$ centered at $\frac{5\pi}{4}$.
   (b) Use Taylor's theorem to find an interval $[\frac{5\pi}{4} - c, \frac{5\pi}{4} + c]$ such that the approximation error of the polynomial you got in part (a) is less than $1/100$ for all x in this interval.

(c) Find the third Maclaurin polynomial of $f(x) = e^{2x}$.
(d) Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial of degree n. Show that the n^{th} Taylor polynomial (centered at x_0) is equal to $f(x)$. (Hint: Use Taylor's theorem.)

(4) (Fourier Polynomials)
(a) Suppose that $f(x)$ is an even function, and $\int_0^\pi f(x)dx = 1$, $\int_0^\pi f(x)\cos x dx = 3$, and $\int_{-\pi}^0 f(x)\cos(2x)dx = 5$. Find the second Fourier polynomial of f.
(b) (i) Show that $\int_{-\pi}^\pi \cos(mx)\cos(nx)dx = 0$, where $n \neq m$ integers. (Hint: Use the identity $2\cos u \cdot \cos v = \cos(u+v) - \cos(u-v)$).
(ii) Evaluate $\int_{-\pi}^\pi \cos^2(mx)dx$.
(iii) Find the fourth Fourier polynomial of $2 + \cos x + 4\cos(2x) + \cos(5x)$.
(c) Find the fourth Fourier polynomial of $f(x) = x^2 + 1$.

(5) (Improper integrals)
(a) Let f be a continuous function. Write down the definition of what it means that $\int_0^\infty f(x)dx = I$.
(b) Evaluate $\int_{-\infty}^\infty \frac{dx}{1+x^2}$.
(c) Evaluate $\int_{-\infty}^\infty e^{-|x|}dx$.
(d) Evaluate $\int_{-1}^\infty x^{-2}dx$.
(e) Show that $\int_1^\infty \frac{dx}{x^4+2}$ converges.
(f) Show that $\int_1^\infty \frac{dx}{2x-1}$ diverges.

(6) (Sequences)
(a) Exercises 19-24, section 11.1.
(b) Let $a_1 = 1$, $a_{n+1} = \frac{n}{n+1} \cdot a_n$.
(i) Show that this sequence is bounded and monotone.
(ii) Use part (a) to show that it converges.
(iii) Evaluate the limit of the sequence (hint: write down the first few terms, and look for a pattern.)
(c) Find the limit of the sequence $\int_k^\infty \frac{dx}{1+x^2}$.
(d) Find the limit of the sequence $a_k = \sqrt{k^2+1} - k$.

(7) (Series)

(a) Evaluate the following series:

(i) $\sum_{k=4}^{\infty} 5^{-k}$.

(ii) $\sum_{k=0}^{\infty} 5^{k}$.

(iii) $\sum_{k=1}^{\infty} \frac{1}{(k+2)(k+3)}$.

(b) Determine whether the following series converge/diverge in the specified number of ways:

(i) $\sum_{k=1}^{\infty} \frac{1}{2k^2+1}$. (2 ways)

(ii) $\sum_{k=0}^{\infty} \frac{1}{3^k+2}$. (2 ways)

(iii) $\sum_{k=1}^{\infty} \frac{1}{k}$. (1 way)

(iv) $\frac{4}{7^{10}} + \frac{4}{7^{12}} + \frac{4}{7^{14}} + \frac{4}{7^{16}} + \dots$ (3 ways).

(v) $\sum_{k=1}^{\infty} \frac{\arctan k}{1+k^2}$ (1 way)