# SOLUTIONS TO REVIEW FOR EXAM 3

**Note:** Some solutions are not quite as detailed as I would want to see on the exam. For example, if I say, "converges by the alternating series test," I would expect you to show that the hypotheses of the alternating series test are satisfied (i.e. the signs alternate, the sequence of absolute values of terms is decreasing and its limit is zero). When I ask for an example, I won't provide a solution, but I'm more than happy to tell you if yours is correct.

(1) Which of the following series converge? Fully justify your answer:

(a) $\sum_{k=1}^{\infty}(-1)^{k+1}/k$
**Solution:**Converges by the alternating series test.

(b) $\sum_{k=1}^{\infty} 1/k$
**Solution:**Diverges by integral test (or "$p$-test").

(c) $\sum_{k=1}^{\infty}(-1)^{k+1}\sin k$
**Solution:**Diverges, since $\lim_{k\to\infty}(-1)^{k+1}\sin k \neq 0$.

(d) $\sum_{k=1}^{\infty}(-1)^{k+1}/(k^{1/3})$
**Solution:**Converges by the alternating series test.

(e) $\sum_{k=1}^{\infty} 1/\sqrt{k}$
**Solution:**Diverges by the $p$-test.

(2) Give an example of a series that converges, but does not converge absolutely.
**Solution:**Many answers are possible. I'll be more than happy to tell you if your example is correct.

(3) Given that $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$, find $\sum_{k=0}^{\infty} 1/(2k+1)^2$. (Hint: Evens and odds.)
**Solution:**

$$\sum_{k=1}^{\infty} 1/(2k+1)^2 = \sum_{k=1}^{\infty} 1/k^2 - \sum_{k=1}^{\infty} 1/2k^2 = \frac{\pi^2}{6} - \frac{1}{2^2}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{8}$$

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(4) For a power series centered at 0, which of the following situations are possible? If possible, give an example of a power series with this property. If not, explain why.

(a) Does not converge at 0.
**Solution:**Not possible. The value of the power series at 0 is just the first coefficient.

(b) Converges only at 0.
**Solution:**This is possible. Give an example.

(c) Converges at 7 but does not converge at $-10$.
**Solution:** This is possible. Give an example.

(d) Has interval of convergence:

(i) $(-4, 5]$
**Solution:** Impossible. If it converges at 5, then its radius of convergence must be at least 5. But then it must converge at $-4$.

(ii) $(-4, 4)$
**Solution:** Possible. Give an example. (Hint for this problem: look at the homework from section 11.6.)

(iii) $[-4, 4)$
**Solution:** Possible.

(iv) $(-4, 4]$
**Solution:** Possible.

(v) $[-4, 4]$
**Solution:** Possible.

(5) Show that $f(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$ satisfies the IVP $f'(x) = 2x \cdot f(x)$, $f(0) = 1$. Use this to evaluate $f$.
**Solution:** Clearly $f(0) = 1$. Also:

$$\begin{aligned} f'(x) &= \frac{d}{dx} \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} \\ &= \sum_{k=1}^{\infty} (2k) \frac{x^{2k-1}}{k!} \\ &= 2x \sum_{k=1}^{\infty} \frac{x^{2k-2}}{(k-1)!} \\ &= 2x \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} = 2xf(x) \end{aligned}$$

Since solutions to IVP's are unique, we can solve the IVP by separation of variables to get that $f(x) = e^{x^2}$.

(6) (a) Find the power series of $1/(2 + x)$ centered at 0 by considering the power series for $1/(1 - x)$, and find its interval of convergence.
**Solution:** We have:

$$\begin{aligned} \frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k, \text{ thus:} \\ \frac{1}{1+x} &= \sum_{k=0}^{\infty} (-1)^k x^k \\ \frac{1}{1+(x/2)} &= \sum_{k=0}^{\infty} (-2)^{-k} x^k \end{aligned}$$

But $1/(1+(x/2)) = 2/(2+x)$, and so:

$$\frac{1}{2+x} = \sum_{k=0}^{\infty} 2^{-k-1}(-1)^k x^k$$

The ratio test tells us that the radius of convergence is 2 (check this), so we need to check the endpoints. At 2, the series becomes (after simplifying) $\sum(-1)^k/2$, and at $-2$, the series becomes $\sum 1/2$, both of which diverge. Thus the interval of convergence is $(-2,2)$.

(b) Use your answer from part (a) to find the power series centered at 0 for $\ln(2+x)$. Find its interval of convergence.

**Solution:**Note that $\int \frac{dx}{2+x} = \ln(2+x)+C_1$. Integrating the series from part (a), we get:

$$\int \left(\sum_{k=0}^{\infty} 2^{-k-1}(-1)^k x^k\right) dx \quad = \quad C_2 + \sum_{k=0}^{\infty} \frac{2^{-k-1}(-1)^k x^{k+1}}{k+1}$$

We can solve for $C_2$ by plugging in 0. We see that $\ln 2 = C_2 + 0$. We find from the ratio test that the radius of convergence is 2, and it converges at 2 (alternating series test), diverges at $-2$ (harmonic series diverges). Thus the interval of convergence is $(-2,2]$.

(c) Use your answer from part (a) to find the power series for $1/(2+x)^2$. Find its interval of convergence.

**Solution:**Since $\frac{d}{dx}\frac{1}{2+x} = -1/(2+x)^2$, we see that:

$$\begin{aligned}
\frac{1}{(2+x)^2} &= -\frac{d}{dx}\frac{1}{2+x} \\
&= -\frac{d}{dx}\sum_{k=0}^{\infty} 2^{-k-1}(-1)^k x^k \\
&= -\sum_{k=1}^{\infty} k2^{-k-1}(-1)^k x^{k-1} \\
&= \sum_{k=1}^{\infty} k(-2)^{k-1}x^{k-1}
\end{aligned}$$

We see that this series diverges at both $-2$, and 2, and the ratio test tells us that this series has radius of convergence 2, so the interval is $(-2,2)$.

(7) Given that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, and $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, find the first four terms of the power series for $e^x/(1-x)$. (Remember, you can multiply power series just like polynomials.)

**Solution:**: Since the product of any terms of degree more than three must have degree more than three, the first four terms of the product is the same as the first four terms of:

$$(1+x+x^2+x^3)\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\right) = 1+2x+\frac{5x^2}{2}+\frac{8x^3}{3}+\frac{5x^4}{3}+\frac{2x^5}{3}+\frac{x^6}{6}$$

So the first four terms are: $1+2x+\frac{5x^2}{2}+\frac{8x^3}{3}$.

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(8) Find the Taylor series of:

(a) $f(x) = e^{2x}$, centered at 0.

**Solution:** $\sum_{k=0}^{\infty} \frac{2^k}{k!} x^k$ (on the exam, you would need to show work justifying this answer)

(b) $f(x) = 2\sin x$, centered at 0.

**Solution:** $\sum_{k=0}^{\infty} \frac{2(-1)^k}{(2k+1)!} x^{2k+1}$

(c) $f(x) = \ln x$, centered at 1.

**Solution:** $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x-1)^k$

(9) Covert the following from polar to rectangular coordinates:

(a) $(\sqrt{2}, \pi/4)$.

**Solution:** $(1, 1)$

(b) $(9, 5\pi/4)$. **Solution:** $(-9/\sqrt{2}, -9/\sqrt{2})$

(c) $r = 1/(1 + \cos\theta)$.

**Solution:** We know that $r = \sqrt{x^2 + y^2}$ and $x = r\cos\theta$, so $\cos\theta = x/\sqrt{x^2 + y^2}$. The equation then becomes:

$$\begin{aligned}
\sqrt{x^2+y^2} &= 1/(1 + x/\sqrt{x^2+y^2}) \\
\sqrt{x^2+y^2}\left(1 + x/\sqrt{x^2+y^2}\right) &= 1 \\
\sqrt{x^2+y^2} + x &= 1 \\
x^2 + y^2 &= (1-x)^2 \\
x^2 + y^2 &= 1 - 2x + x^2 \\
\frac{1-y^2}{2} &= x
\end{aligned}$$

This is a parabola.

(d) $r = 5\sec\theta$.

**Solution:** If we draw our "standard" right triangle, we see that the hypotenuse has length $5/\cos\theta$, so side adjacent to the angle must have length $\cos\theta \cdot (5/\cos\theta) = 5$. In other words, this is the line $x = 5$.

(e) $r = 2\csc\theta$.

**Solution:** Again, converting to rectangular coordinates, we see that the graph of this is the set of all points of the form $(2\cot\theta, 2)$, i.e. the line $y = 2$.

(10) Find the slope of the tangent line at $(\pi/4, \pi/4)$ for $r = \theta$. Find the area of this curve for $\theta$ in $[0, \pi]$.

**Solution:** Converting this to rectangular coordinates yields: $(x(\theta), y(\theta)) = (\theta\cos\theta, \theta\sin\theta)$. Then:

$$(x'(\theta), y'(\theta)) = (\cos\theta - \theta\sin\theta, \sin\theta + \theta\cos\theta)$$

Plugging in $\pi/4$, we get $(x', y') = \frac{\sqrt{2}}{2}(1 - \frac{\pi}{4}), \frac{\sqrt{2}}{2}(1 + \frac{\pi}{4})$. Thus the slope of the tangent line is $y'/x' = (1 + \frac{\pi}{4})/(1 - \frac{\pi}{4})$.

The area of the region swept by this curve is $\int_0^{\pi} \frac{\theta^2 d\theta}{2} = \frac{\pi^3}{6}$.

(11) A particle's position at time $t$ is described by $p(t) = (x(t), y(t))$. $p(0) = (1, 1)$, $p'(0) = (2, 3)$, and $p''(t) = (1, t)$. Find expressions for the velocity and position functions. What is the speed of this particle at time $t = 4$?

**Solution:**$p'(t) = \int p''(1,t)dt = (t, t^2/2) + C_1$. By plugging in 0, we see that $C_1 = (2, 3)$. Also:

$$p(t) = \int p'(t)dt = C_2 + \left(\frac{t^2}{2} + 2t, \frac{t^3}{6} + 3t\right)$$

Solving for $C_2$, we see that $p(t) = \left(\frac{t^2}{2} + 2t + 1, \frac{t^3}{6} + 3t + 1\right)$

The speed of the particle is thus $|p'(4)| = |(4 + 2, 4^2/2 + 3)| = \sqrt{6^2 + (11/2)^2} \approx 8.1$.