NOTES ON COMPLETING THE SQUARE

Completing the square is a very simple technique from high school algebra. The idea is to write a quadratic polynomial $ax^2 + bx + c$ as a square plus a constant (here we assume $a > 0$):

$$\begin{aligned} ax^2 + bx + c &= (dx + e)^2 + f \\ &= d^2x^2 + 2dex + (e^2 + f) \end{aligned}$$

We can easily solve for d, e and f by equating coefficients above, giving three equations in the unknowns d, e and f:

$$\begin{aligned} a &= d^2 \\ b &= 2de \\ c &= e^2 + f \end{aligned}$$

Thus $d = \sqrt{a}$, and so $e = \frac{b}{2d} = \frac{b}{2\sqrt{a}}$, and $f = c - \frac{b^2}{4a}$.

Example:

$$\begin{aligned} 25x^2 - 10x - 3 &= (\sqrt{25}x + \frac{-10}{2\sqrt{25}})^2 + (-3 - \frac{10^2}{4 \cdot 25}) \\ &= (5x - 1)^2 + (-3 - 1) \\ &= (5x - 1)^2 - 4 \end{aligned}$$

It's probably best not to memorize a formula: just remember how the formula was derived. You can always figure out what $dx + e$ is by just looking at the first two coefficients, and then the constant term is "whatever's left" to make the two sides equal.

Example: The quadratic formula. We are solving the equation $ax^2 + bx + c = 0$. Again, we can assume that $a > 0$ (if not, then we can multiply by -1 to make it bigger than 0). Then we have:

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 + \left(c - \frac{b^2}{4a}\right) &= 0 \\
\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 &= \frac{b^2}{4a} - c = \frac{b^2 - 4ac}{4a} \\
\sqrt{a}x + \frac{b}{2\sqrt{a}} &= \pm\sqrt{\frac{b^2 - 4ac}{4a}} \\
\sqrt{a}x &= \frac{-b}{2\sqrt{a}} \pm \frac{\sqrt{b^2 - 4ac}}{2\sqrt{a}} \\
x &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$