Lab 11

April 22, 2009

1. Given a polar equation $r = f(\theta)$, what properties of f determine whether it is symmetric with respect to the x-axis, the y-axis, or the origin? Give examples of each of these types of symmetry. Under what conditions does the presence of two types of symmetry imply the third?

2. (a) Convert the polar equation $r = f(\theta)$ to a rectangular function $\mathbf{g}(\theta) = (x(\theta), y(\theta))$. (Hopefully you're getting good at this by now.)

   (b) Use this and the arc length formula we found last week to show that the arc length of $r = f(\theta)$ for $\alpha \leq \theta \leq \beta$ is given by:

$$S = \int_{\alpha}^{\beta} \sqrt{(f'(\theta))^2 + (f(\theta))^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

   (c) Use the formula from (b) to find the circumference of a circle of radius R centered at the origin. Check that this agrees with the usual formula.

   (d) Find the length of the curve $r = e^{\theta}$ for $-\ln 3 \leq \theta \leq 0$.

3. Let $\mathbf{f}(t) = (x(t), y(t))$ be a vector valued function. Define the unit tangent vector at t to be $\mathbf{T}(t) = \dfrac{\mathbf{f}'(t)}{|\mathbf{f}'(t)|}$. Note that $|\mathbf{T}(t)| = 1$ for all t (unit length, and that $\mathbf{T}(t)$ is tangent to the curve defined by $\mathbf{f}$ at the point $\mathbf{f}(t)$. We can think of $\mathbf{T}$ as a vector that is points where $\mathbf{f}$ is going, but doesn't care how fast $\mathbf{f}$ is getting there.

   Define the *curvature* at t to be $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{f}'(t)|}$. Think of this as a measure of how quickly f is changing direction.

   (a) Find the curvature of a circle of radius R at t, parameterized by $\mathbf{f}(t) = (R\cos t, R\sin T)$. (Hint: your answer should not depend on t.)

   (b) What do you think the curvature of a line should be? Explain.

   (c) Find the curvature of a line $\mathbf{f}(t) = (a, b) + t(c, d)$ at t.

   (d) Draw a graph of $f(t) = (t, t^2)$. Where do you think the curvature is biggest? Compute the curvature and to check your guess.

   (e) Find a formula to compute the curvature of the polar curve $r = g(\theta)$.