SOLUTIONS TO LAB 1

(1) Determine the following limits:
  (a) $\lim_{x\to 5} x^2+1$
  (b) $\lim_{x\to\infty} \sin x$
  (c) $\lim_{x\to 2} \dfrac{x+2}{x^2-4}$
  (d) $\lim_{x\to -3} \dfrac{x+3}{x^2-9}$
  (e) $\lim_{x\to\infty} 2^{-x}$

  Solution:
  (a) Since x^2+1 is a continuous function, we can just substitute: $5^2+1=26$.
  (b) This limit does not exist. To see this, note that $\sin x = 1$ for $x = 2k\pi + \pi/2$ and $\sin x = -1$ for $x = 2k\pi - \pi/2$. Thus if this limit exists, it would have to be close to both 1 and -1, which is impossible.
  (c) This limit also does not exist. We can factorize the denominator as $x^2-4=(x+2)(x-2)$, so the limit is equal to $\lim_{x\to 2} \dfrac{1}{x-2}$. Since the numerator has limit 1, and the denominator has limit 0, the limit as a whole does not exist.
  (d) Again we can factorize the denominator as $x^2-9=(x-3)(x+3)$, so the limit is equal to $\lim_{x\to -3} \dfrac{1}{x-3}$, which is continuous at -3, so we can substitute. Thus the answer is $\frac{1}{6}$.
  (e) Since the function 2^{-x} has a horizontal asymptote of $y=0$ (as can be seen by looking at its graph), the limit is 0. Another way to think about this is that for any number $\epsilon > 0$, 2^-x is eventually between 0 and ϵ for sufficiently large x. Thus by the definition of limits, the limit must be 0.

(2) Find the derivatives of the following functions with respect to x:
  (a) $\cos x$
  (b) $\sin x$
  (c) $\tan^{-1} x$
  (d) $5x^7+2x^3-4x+3$
  (e) $\frac{x^2+1}{x^2-1}$
  (f) $(x^2+2x+7)^{20}$
  (g) $\sin^2 x + \cos^2 x$
  (h) 10^x
  (i) $\ln(x-1)$
  (j) $\ln e^x$
  (k) $x^{(\sin^2 x)}$
  (l) $(x^2-5x)(\cos x)$.

  Solutions:
  (a) $-\sin x$
  (b) $\cos x$
  (c) $\frac{1}{1+x^2}$
  (d) $35x^6+6x^2-4$ (power rule)
  (e) $\dfrac{(2x)(x^2-1)-(x^2+1)(2x)}{(x^2-1)^2}$ (quotient rule)
  (f) $20(x^2+2x+7)^{19}(2x+2)$ (chain rule).

(g) Since $\sin^2 x + \cos^2 x = 1$, this is just finding the derivative of the constant function 1, which is 0. Alternatively, we can use the chain rule to get $2\sin x\cos x + 2(\cos x)(-\sin x) = 0$.
(h) We have that $10^x = (e^{\ln 10})^x = e^{(\ln 10)x}$, so by the chain rule, the derivative is $(\ln 10)e^{(\ln 10)x)} = (\ln 10)10^x$.
(i) Since $d/dx(x-1) = 1$, the chain rule gives us that the derivative is $\frac{1}{x-1}$.
(j) Since natural log and the exponential function are inverses of each other, we have that $\ln e^x = x$, which clearly has derivative 1. Otherwise, we can use the chain rule to determine that the derivative is $\frac{(e^x)'}{e^x} = 1$.
(k) Note that $x^{(\sin^2 x)} = (e^{\ln x})^{\sin^2 x} = e^{\ln x \sin x}$. Now we use the chain rule to get that the derivative is:

$$(\frac{d}{dx}\ln x\sin^2 x)e^{\ln x \sin^2 x} = (\frac{\sin^2 x}{x} + (\ln x)(2\sin x)\cos x)x^{\sin^2 x}$$

(l) $(2x-5)\cos x + (x^2-5x)(-\sin x)$. (product rule)

(3) Find the antiderivatives of the following functions with respect to x:
(a) $t^2 \cdot \cos x$
(b) $\sin x \cdot \cos x$
(c) $x^2 + 2x + 3$
(d) 5
(e) $e^{x+1} + 1$
(f) $x^{-1} + x$
(g) $\frac{2}{1+x^2}$
(h) $f'(x)/f(x)$.

Solutions:
(a) Since t is constant (with respect to x), we have that $\int t^2 \cos x dx = t^2 \int \cos x dx = t^2 \sin x + C$.
(b) Let $u = \sin x$, so that $du = \cos x dx$. Then $\int \sin x \cos x dx = \int u du = u^2/2 + C = \cos^2 x/2 + C$. (integration by substitution)
(c) $\int (x^2 + 2x + 3)dx = \frac{x^3}{3} + \frac{2x^2}{2} + 3x + C$.
(d) Let $u = x+1$, so that $du = dx$. Then $\int (e^{x+1}+1)dx = \int e^u du + \int 1 dx = e^u + x + C = e^{x+1} + x + C$. (integration by substitution)
(e) $\int (x^{-1} + x)dx = \int x^{-1}dx + \int x dx = \ln|x| + x^2/2 + C$.
(f) $\int \frac{2dx}{1+x^2} = 2\int \frac{dx}{1+x^2} = 2\tan^{-1} x + C$.
(g) Let $u = f(x)$, so that $du = f'(x)dx$. Then $\int \frac{f'(x)dx}{f(x)} = \int \frac{du}{u} = \ln|u| + C = \ln|f(x)| + C$. (integration by substitution)

(4) Find the (signed) area under the following functions on the corresponding intervals:
(a) $x^2 - 2x + 1$ on $[0, 1]$
(b) $\sin^3 x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (*Hint*: think about the geometry or use a trig identity)
(c) $\frac{2x}{x^2+1}$ on $[1, 2]$.

Solutions:
(a) $\int_0^1 (x^2 - 2x + 1)dx = [x^3/3 - x^2 + x]_0^1 = 1/3 - 1 + 1 - (0 - 0 + 0) = -2/3$. (fundamental theorem of calculus)
(b) Since sin is an odd function (i.e. $\sin(-x) = -\sin x$), so is $\sin^3$. Thus the area on $[-\frac{\pi}{2}, 0]$ is the negative of the area on $[0, \frac{\pi}{2}]$, so the area is zero. Alternatively, we can use the identity $\sin^2 x + \cos^2 x = 1$ or equivalently $\sin^2 x = 1 - \cos^2 x$. Thus:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin x)(1 - \cos^2 x)dx.$$

Now we use the substitution $u = \cos x$ to get:

$$\int_{\cos(-\frac{\pi}{2})}^{\cos\frac{\pi}{2}} (1 - u^2)du = \int_0^0 (1 - u^2)du = 0$$

(c) Let $u = x^2 + 1$, so $du = 2x$. Thus $\int_1^2 \frac{2x dx}{x^2+1} = \int_{1^2+1}^{2^2+1} \frac{du}{u} = [\ln|u|]_2^5 = \ln 5 - \ln 2$

(5) Describe (using appropriate calculus terms) the relationship of the position $x(t)$, and acceleration $a(t)$ of an ant moving along the x-axis to its velocity $v(t)$.

Solution: Velocity is the rate of change of position (i.e. the derivative of position), and acceleration is the rate of change of velocity (i.e. the derivative of velocity. Also note that position is an antiderivative of velocity, and velocity is an antiderivative of acceleration.

(6) State the following rules in your own words/symbols, and ask me to look at your answers when you're done. If your group is uncertain about one of the rules, ask. Chain rule, product rule, quotient rule, L'Hôpital's rule, integration by substitution, fundamental theorem of calculus.

Solution:

(a) *Chain rule.* $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$. A surprisingly large number of groups got this wrong, so make sure you understand it.

(b) *Product rule.* $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$.

(c) *Quotient rule.* $\frac{d}{dx} f(x)/g(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

(d) *L'Hôpital's rule.* A limit $\lim_{x\to a} f(x)/g(x)$ is of *indeterminate form* if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, or $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$. In this case, $\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f'(x)/g'(x)$.

(e) *Integration by substitution:*

(i) $\int g(f(x))f'(x)dx = \int g(u)du$, where $u = f(x)$.

(ii) $\int_a^b g(f(x))f'(x)dx = \int_{f(a)}^{f(b)} g(u)du$.

(f) *Fundamental Theorem of Calculus:*

(i) If F is an antiderivative of f, then $\int_a^b f(x)dx = F(b) - F(a)$. In words "the area under the curve $y = f(x)$ on the interval $[a, b]$ is $F(b) - F(a)$."

(ii) The function $F(x) = \int_a^x f(t)dt$ is an antiderivative of f. In other words, $F' = f$.

Note: Taken together these statements amount to roughly that "finding area" and differentiation are inverse operations.