Lab 6

Improper Integrals

1. (Fourier polynomial reinforcement)
   Let $k > 0$ be an integer. Recall that $\sin(\pm k\pi) = 0$, $\cos(\pm k\pi) = 1$ if k is even, $\cos(\pm k\pi) = -1$ if k is odd.
   (a) Evaluate $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3 dx$.
   (b) Evaluate $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \cos(kx) dx$ using integration by parts or by noting that x^3 is an odd function.
   (c) Evaluate $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin(kx) dx$ by using integration by parts.
   (d) Find the fourth Fourier polynomial $q_4(x)$ of $f(x) = x^3$. Graph $y = q_4(x)$ and $y = x^3$ on the interval $[-\pi, \pi]$.

2. We need to break up some improper integrals because if we don't, we get inconsistent answers depending on how we take the limit. Note how the first two expressions yield different values.
   (a) Evaluate $\lim\limits_{t\to\infty} \displaystyle\int_{-\sqrt{t}}^{\sqrt{t+2}} x dx$
   (b) Evaluate $\lim\limits_{t\to\infty} \displaystyle\int_{-t}^{t} x dx$.
   (c) Evaluate $\int_{-\infty}^{\infty} x dx$. (Hint: do not use parts (a) or (b).)

3. Consider the improper integral
$$I = \int_{-\infty}^{\infty} \frac{1}{x^3}\, dx$$
   (a) Describe every way in which this integral can be considered improper.
   (b) Rewrite this integral as the sum of integrals over the intervals $(-\infty, -a]$, $[-a, 0]$, $[0, a]$, and $[a, \infty)$ where $a > 0$.
   (c) Rewrite the integrals in (b) as limits of proper integrals.
   (d) Using the fact that $1/x^3$ is an odd function, evaluate all four of these integrals.

4. Consider the improper integral
$$I = \int_{1}^{\infty} \frac{1}{x^2}\, dx$$
   (a) Find a value of b such that
$$\left|1 - \int_{1}^{b} \frac{1}{x^2}\, dx\right| \leq 10^{-8}$$
   (b) Find a value of b such that
$$\int_{b}^{\infty} \frac{1}{x^2}\, dx \leq 10^{-8}$$

5. Consider the integrals $I = \int_0^1 \frac{dx}{\sqrt{x}}$ and $J = \int_0^1 x\, dx$.
   (a) Evaluate I and J.
   (b) Perform the substitution $u = 1/x$ on both I and J. Rewrite the integrals in terms of u. Evaluate these integrals.