

## Lab 7: Sequences and series

March 11, 2009

1. (a) Show that  $(1-x)(x^n + x^{n-1} + x^{n-2} + \cdots + x + 1) = 1 - x^{n+1}$ .  
(b) Show that  $\sum_{k=0}^n ar^k = a \cdot \frac{1-r^{n+1}}{1-r}$ .
2. Let  $f(x) = e^x$ .
  - (a) Compute the first, second, third, fourth and fifth Maclaurin polynomial for  $f(x)$ .
  - (b) See a pattern? Give an expression (in  $\sum$  notation) for the  $n^{\text{th}}$  Maclaurin polynomial of  $f$ .
  - (c) Explain why  $|f^{(n+1)}(x)| \leq 3$  on the interval  $[0, 1]$ . (Hint:  $e < 3$ .)
  - (d) Use Taylor's theorem to evaluate  $\sum_{k=0}^{\infty} \frac{1}{k!}$ . (Hint: use parts (b) and (c), with  $x = 1$ .)
3. Let  $H_n = \sum_{k=1}^n \frac{1}{k}$ , and let  $I_n = \int_1^{n+1} \frac{dx}{x}$ .
  - (a) Evaluate  $I_n$
  - (b) Write an expression for  $L_3$  and  $R_3$  of  $I_3$ . (Left and right Riemann sums. Remember those?)
  - (c) Generalize your answer to (a) to write an expression for  $L_n$  and  $R_n$  of  $I_n$ . (Hint: We are dividing  $[1, n+1]$  into  $n$  subintervals, so  $\Delta x = 1$ , and  $x_i = 1 + i$ .)
  - (d) Show that  $f(x) = \frac{1}{x}$  is decreasing on the interval  $[1, \infty]$ .
  - (e) Use part (d) to write an inequality relating  $L_n, R_n$  and  $I_n$ .
  - (f) Show that  $R_n = L_n - 1 + \frac{1}{n+1}$ .
  - (g) Note that  $H_n = L_n$ . Express your answer from (e) in terms of  $H_n$ , and use this to show that  $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$
  - (h) Show that  $\ln(n+1) \leq H_n \leq \ln(n+1) + 1 - \frac{1}{n+1}$ .

4. Consider the series  $\zeta(p) = \sum_{k=1}^{\infty} \frac{1}{k^p}$ , where  $p > 1$  is a fixed constant. This is called the Riemann Zeta function, which is very important in number theory. We will show that it converges.

(a) Let  $S_n = \sum_{k=1}^n \frac{1}{k^p}$  (the  $n^{\text{th}}$  partial sum). Show that the sequence of partial sums is increasing.

(b) Show that  $S_{2m+1} = 1 + \sum_{k=1}^m \frac{1}{(2k)^p} + \sum_{k=1}^m \frac{1}{(2k+1)^p}$ . (Hint: the first sum is just a sum over the even indices, the second is a sum over the odd indices.)

(c) Show that  $\frac{1}{(2k+1)^p} < \frac{1}{(2k)^p}$ . (Hint:  $1/(x+1) \leq 1/x$ .)

(d) Use part (c) to show that  $\sum_{k=1}^m \frac{1}{(2k+1)^p} < \sum_{k=1}^m \frac{1}{(2k)^p}$ , and so by part (b):

$$S_{2m+1} \leq 1 + 2 \cdot \sum_{k=1}^m \frac{1}{(2k)^p} = 1 + 2^{1-p} S_m$$

(e) Use parts (d) and (a) to show that  $S_{2m+1} < 1 + 2^{1-p} S_{2m+1}$ .

(f) Show that  $\{S_m\}$  is bounded above (Hint: Move all the  $S_{2m+1}$  terms to the left side, find a common factor, and divide.) Use this and part (a) to show that  $\sum_{k=1}^{\infty} \frac{1}{k^p}$  converges.