Lab 8: Testing for Convergence

March 18, 2009

1. (Warm up) Suppose that $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ converge. Show the following by looking at partial sums and using basic properties of limits:

   (a) $\sum_{k=0}^{\infty}(a_k + b_k) = \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$.

   (b) $\sum_{k=0}^{\infty} c \cdot a_k = c \cdot \sum_{k=0}^{\infty} a_k$

   (c) If $a_k < b_k$ for all k, then $\sum_{k=0}^{\infty} a_k \leq \sum_{k=0}^{\infty} b_k$.

2. The ratio test is essentially a special case of the comparison test.

   (a) Suppose that $a_0 = 1$, $a_n > 0$ for all n, and $a_{n+1}/a_n \leq 1/2$ for all n. Show that $a_n \leq 2^{-n}$ for all n. Hint: $\frac{a_n}{a_0} = \frac{a_1}{a_0} \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdots \frac{a_n}{a_{n-1}}$.

   (b) Use the comparision test to show that the series $\sum_{n=0}^{\infty} a_n$ converges.

   (c) Now suppose that $b_n > 0$ for all n, and $b_{n+1}/b_n \leq r$ for all n. Show that $b_n \leq a_0 \cdot r^n$. (Hint: Use the hint from part (a))

   (d) Show that if $r < 1$, then $\sum_{n=0}^{\infty} b_n$ converges.

   (e) Now suppose that $c_n > 0$ for all n, and $\lim_{n\to\infty} c_{n+1}/c_n = r_0$, where $0 \leq r_0 < 1$ is a constant. Then eventually $c_{n+1}/c_n < \frac{r_0+1}{2} = r$ for all sufficiently large n (say for all $n \geq N$ for some sufficiently large N), because otherwise the sequence c_{n+1}/c_n is far away from r_0 infinitely often, which contradicts that $\lim c_{n+1}/c_n = r_0$.

   Show that $\sum_{n=0}^{\infty} c_n$ converges. (Hint: Show that the tail $\sum_{n=N}^{\infty} c_n$ converges by using part (d) with $b_n = c_{N+n}$.)

3. (Telescoping series.) The idea here is to find a simple expression for the partial sums, and use that to evaluate the actual value of the partial sum.

   (a) Find the partial fractions decomposition of $\frac{1}{x^2-1}$, and use it to evaluate $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$. (Hint: look at the partial sums. A lot of terms should cancel.) Show that this series converges in two other ways.

   (b) i. Use integration by parts to evaluate $\int \arctan x dx$

      ii. Show that $f(x) = \arctan(x+1) - \arctan x$ is decreasing and positive on $[0, \infty)$. (Hint: To show f is positive, first show that arctan is an increasing function.)

      iii. Use the integral test to show that $\sum_{k=0}^{\infty}(\arctan(k+1) - \arctan k)$ converges.

      iv. Find a simple formula for $\sum_{k=0}^{n}(\arctan(k+1) - \arctan k)$. (Hint: all but two terms should cancel.)

      v. Evaluate $\sum_{k=0}^{\infty}(\arctan(k+1) - \arctan k)$.