Lab 9: A series of serious facts about series

April 8, 2009

Note: I think that today went better than expected, so I'm changing the reading assignment to 11.7, and we'll work on 11.7 next time.

1. **A proof of the alternating series test.** Let $S_n = \sum_{k=1}^{\infty} (-1)^{k+1} c_k$, where $c_1 \geq c_2 \geq c_3 \geq \cdots \geq 0$, and $\lim_{k\to\infty} c_k = 0$.
   (a) Show that the sequence of even partial sums $S_2, S_4, S_6, \ldots$ is increasing.
   (b) Show that the sequence of odd partial sums $S_1, S_3, S_5, \ldots$ is decreasing.
   (c) Show that $S_{2m} \leq S_{2m-1}$ for all $m \geq 1$.
   (d) Use part (c) to show that the sequence of even partial sums and the sequence of odd partial sums both converge. (Note that we don't yet know they have the same limit.)
   (e) Show that $\lim_{m\to\infty} (S_{2m+1} - S_{2m}) = 0$. From this it follows that $\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} (-1)^{k+1} c_k$ converges to some real number S.
   (f) Explain why $0 < S - S_{2m} < c_{2m+1}$ and $0 < S_{2m+1} - S < c_{2m+2}$.

2. Suppose that $f(x) = \sum_{k=0}^{\infty} a_k x^k$ converges on the interval $(-\infty, \infty)$.
   (a) Find $f'(0)$, $f''(0)$ and $f'''(0)$. See a pattern? Find an expression for $f^{(n)}(0)$.
   (b) Write down the expression for the n^{th} Maclaurin polynomial of a function $g(x)$.
   (c) Write down the expression for the n^{th} Maclaurin polynomial $p_n(x)$ of $f(x)$ using your answer for part (a). Simplify.
   (d) Show that $\lim_{n\to\infty} p_n(x) = f(x)$ for all x.

3. (A proof that e is irrational) Assume that e is rational, and $e = m/n$, where m, n are positive integers. Our goal is get a contradiction from these assumptions, showing that e cannot be written as a fraction (i.e. e is irrational).

   Recall that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

   (a) Explain why $m!/e$ is an integer (using the assumption above).

   (b) Explain why $m! \sum_{k=0}^{m} \frac{(-1)^k}{k!}$ is an integer. (Hint: show that $m!/k!$ is an integer first.)

   (c) Show that $N = m! \left| \frac{1}{e} - \sum_{k=0}^{m} \frac{(-1)^k}{k!} \right|$ is an integer.

   (d) Use Taylor's theorem to show that:

   $$N = m! \left| \frac{1}{e} - \sum_{k=0}^{m} \frac{(-1)^k}{k!} \right| \leq m! \cdot \frac{1}{(m+1)!} = \frac{1}{m+1}$$

   Hint: Use part (c) from the previous problem and the fact that $0 < e^x \leq 1$ on the interval $(-\infty, 0]$.

   (e) Explain why parts (c) and (d) show that $N = 0$.

   (f) Explain why (e) is impossible, and so e is not a rational number. (Hint: show that:

   $$0 \neq \sum_{k=m+1}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{e} - \sum_{k=0}^{m} \frac{(-1)^k}{k!}$$

   by looking at the bounds given by the alternating series test.)