WORKSHEET FOR 1/23/2009 - SOLUTIONS

(1) Suppose that $\int_0^x f(t)dt = 3x^2 + e^x - \cos x$. Find $f(2)$.
Solution: We must use a version of the fundamental theorem of calculus, that states that:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

With this in mind, we can differentiate both sides of the equation given in the problem to get:

$$\begin{aligned}
\frac{d}{dx}\int_0^x f(t)dt &= \frac{d}{dx}(3x^2 + e^x - \cos x) \\
f(x) &= \frac{d}{dx}(3x^2 + e^x - \cos x) \text{ (by fundamental theorem of calculus)} \\
f(x) &= 6x^2 + e^x + \sin x \\
f(2) &= 24 + e^2 + \sin 2.
\end{aligned}$$

(2) A valve is opened, and water starts flowing through a pipe at time $t = 0$. Suppose that $f(t)$ gallons/second of water are flowing through a pipe for any time $t \geq 0$. Write down an expression for the amount of water than flows through the pipe in its first 30 seconds of operation.
Solution: $\displaystyle\int_0^{30} f(t)dt.$

(3) Let $F(x) = \int_0^x (\cos t)^2 e^t dt$.
   (a) Find the critical numbers of F.
   (b) Where is F concave up and concave down? Find local maxima and minima on the interval $[-1, 2\pi]$?
   (c) Find the where F achieves its maximum and minimum values on the interval $[-1, 2\pi]$.

Solution:
   (a) By the fundamental theorem of calculus:

$$F'(x) = \frac{d}{dx}\int_0^x (\cos t)^2 e^t dt = (\cos x)^2 e^x$$

   Since $e^x \neq 0$, this function has critical numbers whenever $\cos x = 0$, i.e. for $x = \frac{\pi}{2} + k\pi$, where k is an integer.
   (b) Omitted.
   (c) Since $(\cos t)^2 e^t \geq 0$, cumulative area must be increasing. Thus $F(x)$ never decreases, so the minimum must occur at the left enpoint -1, and the maximum must occur at the right endpoint 2π.

(4) Show that $0 \leq \int_0^\pi \sin^2 x dx \leq \pi$. *Hint.* You will not be able to compute the actual area. Draw a graph and think about the area.
Solution: Since $0 \leq \sin^2 x \leq 1$, we have that $\int_0^\pi 0 dx \leq \int_0^\pi \sin^2 x dx \leq \int_0^\pi 1 dx$, and

the inequality follows immediately. Geometrically, what we are doing is noticing that the graph of $y = \sin^2 x$ on $[0, \pi]$ is completely contained inside a $1 \times \pi$ rectangle, so it must have less area than this rectangle.

(5) Evaluate $\int_0^{2\pi} |\sin x| dx$.
Solution: First note that:

$$|\sin x| = \begin{cases} \sin x & \text{if } 0 \leq x \leq \pi \\ -\sin x & \text{if } \pi < x \leq 2\pi \end{cases}$$

Also recall that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$. Thus we have

$$\begin{aligned} \int_0^{2\pi} |\sin x| dx &= \int_0^{\pi} |\sin x| dx + \int_{\pi}^{2\pi} |\sin x| dx \\ &= \int_0^{\pi} \sin x dx + \int_{\pi}^{2\pi} -\sin x dx \\ &= [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi} \\ &= (1 - (-1)) + (1 - (-1) = 4 \end{aligned}$$

(6) Evaluate $\int_{-1}^{1} \sqrt{1 - x^2} dx$. *Hint.* You will probably not be able to find an antiderivative. What shape is this describing?
Solution: The region bounded by the graph of this function and the x-axis is a semicircle of radius 1, so the area is one half of the corresponding circle, i.e. $\frac{1}{2}\pi \cdot 1^2 = \frac{\pi}{2}$.