WORKSHEET FOR 1/26/2009 - SOLUTIONS

(1) Evaluate the following using integration by substitution, and check your answer by differentiating.

(a) $\int \sqrt{3x-2}dx$

(b) $\int_2^1 x\sqrt{3x-2}dx$ (Hint: solve for x in terms of u.)

(c) $\int \frac{x^2}{1+x^6}dx$.

Solutions:

(a) Let $u = 3x - 2$. Then $du = 3dx$, so $du = dx/3$. Thus

$$\begin{aligned}\int \sqrt{3x-2}dx &= \int \sqrt{u}\frac{du}{3} \\ &= \frac{1}{3}\int \sqrt{u}du \\ &= \frac{1}{3}\cdot\frac{1}{\frac{1}{2}+1}u^{\frac{1}{2}+1} + C \\ &= \frac{2}{9}u^{\frac{3}{2}} + C \\ &= \frac{2}{9}(3x-2)^{\frac{3}{2}} + C\end{aligned}$$

(b) Again, we'll use $u = 3x - 2$. The $du = 3dx$, so $du = dx/3$. We also have that $x = \frac{u+2}{3}$. Thus

$$\begin{aligned}\int_2^1 x\sqrt{3x-2}dx &= \int_{3\cdot 2-2}^{3\cdot 1-2} \frac{u+2}{3}\sqrt{u}\frac{du}{3} \\ &= \frac{1}{9}\int_4^1 (u+2)(u^{\frac{1}{2}})du \\ &= \frac{1}{9}\int_4^1 (u\cdot u^{\frac{1}{2}} + 2u^{\frac{1}{2}})du \\ &= \frac{1}{9}\int_4^1 (u^{\frac{3}{2}} + 2u^{\frac{1}{2}})du \\ &= \frac{1}{9}\left[\frac{2}{5}u^{\frac{5}{2}} + 2\cdot\frac{2}{3}u^{\frac{3}{2}}\right]_4^2 \\ &= \frac{1}{9}\left(\left(\frac{2}{5}1^{\frac{5}{2}} + \frac{4}{3}1^{\frac{3}{2}}\right) - \left(\frac{2}{5}4^{\frac{5}{2}} + \frac{4}{3}4^{\frac{3}{2}}\right)\right) \\ &= \frac{1}{9}\left(\left(\frac{2}{5} + \frac{4}{3}\right) - \left(\frac{2}{5}2^5 + \frac{4}{3}2^3\right)\right) = \frac{-326}{135}\end{aligned}$$

(c) Let $u = x^3$. Then $du = 3x^2dx$, so $du = \frac{dx}{3x^2}$. Thus:

$$\int \frac{x^2}{1+x^6}dx = \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\arctan u + C = \frac{1}{3}\arctan(x^3) + C$$

(2) Consider $\displaystyle\int_{-1}^{3} x^2 dx$ (*).

(a) Evaluate (*) exactly, using the fundamental theorem of calculus.

(b) Approximate (*) by computing L_4. That is, divide the interval $[-1,3]$ into four equal subintervals, and write down the corresponding Riemann sum, sampling at the left endpoints of the interval. You should end up with the sum $\displaystyle\sum_{i=0}^{3} f(x_i)\Delta x_i$, where $f(x) = x^2$, $\Delta x_i = \frac{3-(-1)}{4} = 1$, and $x_i = -1 + i$.

(c) Compute R_4. That is, approximate the integral in the same way as in part (b), but this time use the *right* endpoint of each interval rather than the left.

(d) Compute M_4. Same thing, but using the midpoint of each interval.

(e) Compute T_4. This is just the average of R_4 and L_4. Geometrically, we are using trapezoids instead of rectangles to approximate the area. Which approximation(s) give the best result?

(f) Write down an expression for the approximation R_n to (*). Using this, express (*) as a limit of Riemann sums.

Solutsion:

(a) $\int_{-1}^{3} x^2 dx = [\frac{1}{3}x^3]_{-1}^{3} = \frac{1}{3}(27 - (-1)) = \frac{28}{3} \approx 9.33$.

(b) We break up the interval $[-1,3]$ into the intervals $[-1,0]$, $[0,1]$, $[1,2]$, and $[2,3]$. Here $\Delta x_i = \frac{3-(-1)}{4} = 1$, and we are evaluating at the left endpoints, so $x_0 = -1$ (the left endpoint of the first interval). Thus we have:

$$L_4 = \sum_{i=0}^{3} f(x_0 + i\Delta x_i)\Delta x_i = (-1)^2 + 0^2 + 1^2 + 2^2 = 1 + 1 + 4 = 6.$$

(c) We break up the interval $[-1,3]$ into the same intervals, with the same Δx_i. We are evaluating at the right endpoints, so $x_0 = 0$ (the right endpoint of the first interval). Thus we have:

$$R_4 = \sum_{i=0}^{3} f(x_0 + i\Delta x_i)\Delta x_i = (0)^2 + 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14.$$

(d) We break up the interval $[-1,3]$ into the same intervals, with the same Δx_i. We are evaluating at the midpoints, so $x_0 = -\frac{1}{2}$. Thus we have:

$$\begin{aligned} M_4 &= \sum_{i=0}^{3} f(x_0 + i\Delta x_i)\Delta x_i \\ &= \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2 \\ &= \frac{1}{4} + \frac{1}{4} + \frac{9}{4} + \frac{25}{4} = \frac{36}{4} = 9. \end{aligned}$$

(e) $T_4 = \frac{L_4+R_4}{2} = \frac{14+6}{2} = \frac{20}{2} = 10$. The midpoint approximation gives the best results in this case.

(f) Here $\Delta x = \frac{3-(-1)}{n}$, so we'll break up the interval $[-1,3]$ into the n intervals $[-1+0\cdot\frac{4}{n}, -1+1\cdot\frac{4}{n}]$, $[-1+1\cdot\frac{4}{n}, -1+2\cdot\frac{4}{n}]$, ..., $[-1+(n-1)\frac{4}{n}, -1+n\cdot\frac{4}{n}] = [-1+(n-1)\frac{4}{n}, 3]$. Since we are evaluating at the right endpoints, we can take $x_0 = -1+\frac{4}{n}$ (right endpoint of first interval), and $x_i = x_0 + i\Delta x = -1+\frac{4}{n}+\frac{4i}{n} = -1+\frac{4i+4}{n}$. The general form for the Riemann sum applies here:

$$R_n = \sum_{i=0}^{n-1} f(x_i)\Delta x = \sum_{i=0}^{n-1}\left(-1+\frac{4i+4}{n}\right)^2 \cdot \frac{4}{n}.$$

Finally, $\int_{-1}^{3} x^2 dx = \lim_{n\to\infty} R_n$.

Accuracy: The correct value for the integral is 28/3. We have that $R_{100} - 28/3 \approx 0.161067$ (accurate to about one part in fifty), $L_{100} - 28/3 \approx -0.158933$ (accurate to about one part in fifty), $T_{100} - 28/3 \approx 0.000114286$ (accurate to about one part in 8,000), and $M_{100} - 28/3 \approx -0.000533333$ (accurate to about one part in 17,000).