WORKSHEET FOR 2/9/2009

Reading assignment for Wednesday. Read section 7.4.
Homework due Wednesday. 6, 8, 9abd[1], 18
Notes: Thus far, the numerical methods we've learned essentially consist of assuming that a function is "well-behaved" on small intervals. For example, left and right Riemann sums can be seen as assuming that a function is approximately constant on each interval $[x_i, x_{i+1}]$, trapezoidal and midpoint sums can be seen as assuming that function is linear on this interval, and Simpson's rule can be seen as assuming that a function is quadratic on small intervals. Euler's method is similar, in the sense that we assume that the solution to a differential equation has a constant derivative on small intervals (i.e. is linear).

Say we're given that a function $y(t)$ satisfies the differential equation (DE) $y'(t) = f(y(t), t)$, with initial value $y(a) = y_0$ (this is called an *initial value problem* or IVP). Then we can approximate $y(b)$ as follows:

(1) Break up the interval $[a, b]$ into a bunch of subintervals with endpoints $t_0 = a, t_1, \ldots, t_n = b$, and let $i = 0$.
(2) Set $y_{i+1} = y_i + f(y_i, t_i) \cdot (t_{i+1} - t_i) = y_i + f(y_i, t_i) \cdot \Delta t$. Here we are assuming that $y(t)$ is approximately linear on the interval $[t_i, t_{i+1}]$.
(3) Repeat step (2) with $i := i + 1$, until $i = n$. Then $y_n \approx y(b)$.

Example. $a = 1$, $b = 2$, $y(1) = -1$, $y' = y^2$, with $n = 3$. We divide up $[0, 1]$ into the intervals $[1, \frac{4}{3}]]$, $[\frac{4}{3}, \frac{5}{3}]$ and $[\frac{5}{3}, 2]$. Then:

$y_0 = -1$,
$y_1 = y_0 + (y_0)^2 \cdot (\frac{4}{3} - 1) = -1 + 1 \cdot \frac{1}{3} = \frac{-2}{3}$
$y_2 = \frac{-2}{3} + (\frac{-2}{3})^2 \cdot (\frac{1}{3}) = \frac{-14}{27}$
$y_3 = \frac{-14}{27} + (\frac{-14}{27})^2 \cdot (\frac{1}{3}) \approx -0.4288$.

The actual (exact) solution is $y(t) = -1/t$ (easy to check), which would give $f(2) = -1/2$, which is somewhat close to what we got above. At $n = 50$, we get $y(2) \approx -0.496$, which is closer still.

(1) Repeat the above example with $y' = y^3$.
(2) Consider the IVP $y' = y - 2$, $y(0) = 1$.
    (a) Use Euler's method with 5 steps of size 0.2 to estimate $y(1)$.
    (b) Show that the exact solution of the IVP is $y(t) = 2 - e^t$.
(3) Consider an IVP modeling metabolism of caffeine in the body. Caffeine is metabolized by an enzyme in the liver, and the enzyme only works when a caffeine molecule randomly hits it in the right place. The number of such collisions is proportional to the concentration of caffeine in the blood, so in other words, the amount of caffeine $A(t)$ in the blood follows the following differential equation, known as exponential decay:

$$A' = K \cdot A.$$

Here K is a constant, which depends on the person–in a normal adult, $K \approx \frac{\ln 0.5}{4}$ hours^{-1}.
    (a) Suppose that someone drinks two cups of coffee, for a total of approximately 200 milligrams of caffeine. Use Euler's method with four subdivisions to find approximately how much caffeine is left after four hours.
    (b) Show that $A(t) = A(0) \cdot 2^{-t/4}$ is an exact solution to the IVP.
    (c) Women using birth control can have K values as high as $K \approx \frac{\ln 0.5}{10}$. Repeat part (a) with this value for K.
    (d) Show that $A(t) = A(0) \cdot 2^{-t/10}$ is an exact solution to the IVP in part (c). In other words, the half life of caffeine in this case can be as high as 10 hours in this case.

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[1] Hint: Show that $y_1 = 1 + \frac{1}{n}$, and if $y_i = (1 + \frac{1}{n})^i$, then $y_{i+1} = (1 + \frac{1}{n})^{i+1}$.