WORKSHEET FOR 2/18/2009

Reading assignment for Wednesday. Read section 9.2.
Homework due Friday. 9.1: 13, 14, 19, 34, 36, 37
Instructions for Homework: You do not need to find an interval on which these Taylor series are accurate in 13, 14, 19; An even function is one for which $f(-x) = f(x)$ for all x.
Notes: The idea here is that we are approximating a function with a "best" polynomial of a given degree (Taylor polynomial) around a given point x_0. It turns out that all we need to do to find the best polynomial of degree n is make sure that that the k^{th} derivatives agree for $0 \leq k \leq n$.
Example: In the case of $n = 1$, this is just finding a tangent line (linear approximation):

$$f(x) \approx p_1(x) = f'(x_0) \cdot (x - x_0) + f(x_0)$$

Example: In the case $n = 2$, we want a polynomial $p_2(x)$ such that $p_2(x_0) = f(x_0)$, $p_2'(x_0) = f'(x_0)$, $p_2''(x_0) = f''(x_0)$. This is easiest if we write the polynomial $p_2(x) = a_2(x - x_0)^2 + a_1(x - x_0) + a_0$. Then clearly:
$p_2(x_0) = a_0 = f(x_0)$
$p_2'(x_0) = 2a_2(x_0 - x_0) + a_1 = a_1 = f'(x_0)$
$p_2''(x_0) = 2a_2 = f''(x_0)$, so $a_2 = f''(x_0)/2$.
In other words, $p_2(x) = \frac{1}{2}f''(x_0) \cdot (x - x_0)^2 + f'(x_0) \cdot (x - x_0) + f(x_0)$.
General case: The n^{th} Taylor polynomial of f at x_0 is:

$$p_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} \cdot (x - x_0)^i$$

Here $i! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (i - 1) \cdot i$, and $f^{(i)}$ denotes the i^{th} derivative of f.
Note: Maclaurin polynomials are just Taylor polynomials centered at 0.

(1) Find the fifth order Taylor polynomial of $f(x) = \sin x$ at $x_0 = 0$ and also at $x_0 = \pi$. Graph each of these and $y = \sin x$, and compare the results.
(2) Find the fourth order Maclaurin polynomial of $f(x) = e^x$ at $x_0 = 0$. Use this to approximate $e = e^1$, and find the approximation error. ($e \approx 2.71828183$)
(3) Suppose that $P_5(x) = 3 - x/4 - x^2/5 + 7x^3 - x^4 + 11x^5/9$ is the fifth order Taylor polynomial of f. Find the Taylor polynomial of order 2 for the function f.
(4) Suppose that $P_k(x) = P_i(x)$ for $k, i > n$ are the Taylor polynomials of a function f. What can you say about f.