WORKSHEET FOR 2/27/2009

Reading assignment for Wednesday. Read section 9.3.
Homework due Friday. 9.2: 9cd, 14, 16, 18. Extra credit (5 points) 19. (Note that 9ab are done below in problem 2.)
Notes: Taylor's theorem gives a way of bounding the error of Taylor polynomial approximations to a function.
Theorem. If $|f^{(n+1)}(x)| \leq K_{n+1}$ (i.e. $-K_{n+1} \leq f^{(n+1)}(x) \leq K_{n+1}$) on an interval $[a, b]$. Let $P_n(x)$ be the n^{th} Taylor polynomial centered at x_0. Then we have the following bound on the error of approximation for all x in $[a, b]$:

$$|f(x) - P_n(X)| \leq \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+!}$$

Example: We know that $\sin \frac{\pi}{2} = 1$. Let $P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ be the fifth Maclaurin polynomial for $\sin x$. What is a bound on the approximation error of $P_7(\pi/2)$. We know that any derivative of sin is either $\sin, \cos, -\sin, -\cos$, and each of these are bounded in absolute value by $K_7 = 1$. Thus by Taylor's theorem:

$$|\sin \frac{\pi}{2} - P_7(\frac{\pi}{2})| \leq \frac{1}{(7+1)!}|\frac{\pi}{2} - 0|^{6+1} \approx 0.00091926$$

$P_7(\pi/2) \approx 0.999843$, and $|1 - P_7(\pi/2)| \approx 0.000156899 \leq 0.00091926$.

(1) Let $f(x)$ be a polynomial of degree n. What does Taylor's theorem tell you about the accuracy of the $(n+1)^{\text{th}}$ Taylor polynomial centered at any point? Explain.
(2) Let $f(x) = \sin(x^2)$
   (a) Find the second Maclaurin polynomial of f.
   (b) Use Taylor's theorem to find a bound on the approximation error of P_2 on the interval $[0, 1/2]$.
(3) Use Taylor's theorem (!) to evalue the limit:

$$\lim_{x \to 0} \frac{x - \sin x}{x^2}$$