WORKSHEET FOR 3/2/2009

Reading assignment for Wednesday. Read section 10.1.
Homework due Wednesday. 9.3: 2, 3, 7, 8, 9, 19
History: Fourier polynomials and Fourier series are tremendously important in mathematics, physics and computer science. They were originally developed in Joseph Fourier's solution of the "heat equation" which describes heat dissipation. In mathematics, they are central in harmonic analysis and Hilbert space theory, and the determination of the domains of convergence of Fourier series was also the inspiration for Cantor's development of set theory. In physics, they are used in the solution of a number of physical problems, and since they are central to the theory of separable Hilbert spaces, they are very important in quantum mechanics. In computer science, they are used for audio, video and image compression. Fourier transforms play a key role in MP3 audio compression, which all of you are probably familiar with. As an example, a simple audio compression algorithm might take a short sample of sound, calculate an approximation to a Fourier polynomial, and throw out terms with low coefficients (amplitudes). In other words, we break up the waveform of the sound into a bunch of distinct sound waves, and throw away those with a low "volume." A discrete version of the Fourier transform is used in a variety of algorithms, including algorithms for multiplying large numbers, which are important for scientific computing .
Idea: We want to write a given function f on the interval $[-\pi, \pi]$ as being approximated by a sum of sines and cosines (a Fourier polynomial). In particular, our goal is to write:

$$f(x) \approx q_n(x) = a_0 + a_1 \cos x + a_2 \cos(2x) + \cdots + a_n \cos(nx) + b_1 \sin x + b_2 \sin(2x) + \cdots + b_n \sin(nx)$$

It turns out that the "best" approximation is given by the following formulas for coefficients:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \quad (k \geq 1) \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \quad (k \geq 1) \end{aligned}$$

Useful facts in evaluating coefficients:

- $q_n(x)$ is always periodic, with period 2π.
- If f is an odd function (i.e. $f(-x) = -f(x)$ for all x), then $\int_{-\pi}^{\pi} f(x)\cos(nx)dx = 0$. This means that if f is an odd function, then $a_k = 0$ for all k.
- If f is an even function (i.e. $f(-x) = f(x)$ for all x), then $\int_{-\pi}^{\pi} f(x)\sin(nx)dx = 0$. This means that if f is an even function, then $b_k = 0$ for all k.
- Note that $\sin(nx)$ is an odd function and $\cos(nx)$ is an even function.
- $\int_{-\pi}^{\pi} \cos(kx)dx = 0$, $\int_{-\pi}^{\pi} \sin(kx)dx = 0$
- The Fourier polynomial of a sum is the sum of the Fourier polynomials. More precisely, if $f(x) = g(x) + h(x)$, then the Fourier coefficients of f are the sum of the corresponding coefficients of g and h. Thus you can compute the Fourier expansion of a polynomial by using this idea, the fact that even degree terms are even, odd degree terms are odd, and integration by parts.

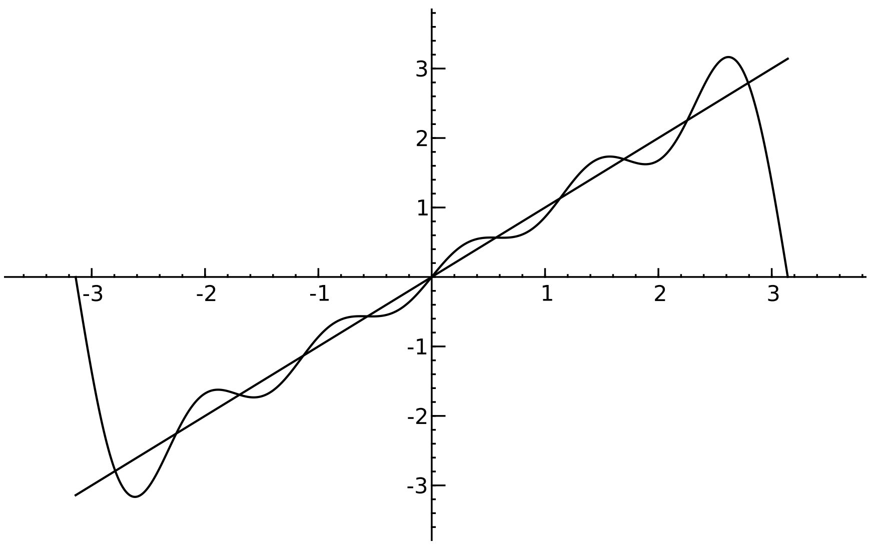
Example: Find the n^{th} Fourier polynomial $q_n(x)$ for the function $f(x) = x$.
Note that since x has odd degree, it is an odd function, so $a_i = 0$ for all i. We compute:

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin kx dx \\ &= \frac{1}{\pi}\left(\left[\frac{-x\cos(kx)}{k}\right]_{-\pi}^{\pi} - \frac{1}{k}\int_{-\pi}^{\pi}(-\cos(kx))dx\right) \quad \text{by integration by parts: } u = x, dv = \sin(kx)dx \\ &= \frac{1}{k\pi}\cdot[-x\cos(kx)]_{-\pi}^{\pi} \end{aligned}$$

We can check that this last expression is $b_k = -\frac{2}{k}$ if k is even, and $b_k = \frac{2}{k}$ if k is odd. Thus:

$$q_n(x) = \sum_{k=1}^{n}(-1)^{k+1}\frac{2}{k}\sin(kx)$$

Graphed below is $y = x$ and $y = q_5(x) = 2\sin x - \sin(2x) + \frac{2}{3}\sin(3x) - \frac{1}{2}\sin(4x) + \frac{2}{5}\sin(5x)$.



Exercises:

(1) Find the third Fourier polynomial for $f(x) = x + 1$. (Hint: use the example above.)
(2) Find the third Fourier polynomial for $f(x) = x^2$. (Hint: x^2 is an even function.)
(3) Combine the first two problems to find the third Fourier polynomial for $f(x) = x^2 - x - 1$.
(4) Find the fourth Fourier polynomial for $f(x) = 2\cos x - 3\sin x + \cos(5x) + \sin(4x)$. (Hint: use exercise 3 on page 515 (from homework).)