WORKSHEET FOR 3/6/2009

Reading assignment for Monday. Read section 11.1. If you will take statistics at some point, reading 10.3 might be a good idea.
Homework due Monday. 10.2: 8, 10, 18, 19, 20, 22
Notes: The comparison test lets you test whether a given integral converges without actually computing it. It can also be used to estimate the values of integrals by using the values of integrals we can compute.

Theorem. *Let f and g be continuous on (a, b) and that $0 \leq f(x) \leq g(x)$ for all x in (a, b). Then*

- *If $\int_a^b g(x)dx$ converges, then so does $\int_a^b f(x)$, and also:*

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

- *If $\int_a^b f(x)dx$ diverges, then so does $\int_a^b g(x)dx$.*

Also useful for dealing with functions that take negative values:

Theorem. *If $\int_a^b |f(x)|dx$ converges, then $\int_a^b f(x)dx$ converges and $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$.*

When dealing with inequalities, it's useful to remember some basics:

- (Triangle inequality) $|f(x) + g(x)| \leq |f(x)| + |g(x)|$
- If $0 < c \leq d$, then $1/c \geq 1/d$.
- If f is an increasing function and $x \leq y$, then $f(x) \leq f(y)$. If f is a decreasing function and $x \leq y$, then $f(x) \geq f(y)$.

Example 1: Determine whether the following integral converges:

$$\int_1^\infty \frac{1}{x^2 - 1} dx$$

Note that $x^2 - 1 \geq x^2/2$ for $x \geq 2$. Thus $\frac{1}{x^2-1} \leq \frac{1}{2x^2}$, and so:

$$\int_2^\infty \frac{dx}{x^2 - 1} \leq \int_2^\infty \frac{2dx}{x^2} = 2 \cdot \int_2^\infty \frac{dx}{x^2}$$

We can show this last integral converges using the methods we learned last time.
Example 2: Determine whether the following integral converges: $\int_1^\infty e^{-x^2} dx$.
First note that $x \leq x^2$ when $x \geq 1$, and e^{-y} is a decreasing function, so $e^{-x^2} \leq e^{-x}$ for all $x \geq 1$. Thus by the comparison test

$$\int_1^\infty e^{-x^2} dx \leq \int_1^\infty e^{-x} dx = \lim_{t \to \infty} \left[-e^{-x}\right]_1^t = 1/e$$

Exercises:

(1) Does $\int_2^\infty \frac{dx}{x-\sqrt{x}}$ converge? Hint: $x-\sqrt{x} \leq x$.

Solution: By the hint, $\frac{1}{x-\sqrt{x}} \geq \frac{1}{x}$. We have that:

$$\int_2^\infty \frac{dx}{x} = \lim_{t\to\infty} [\ln|x|]_2^t = \infty$$

Thus by the comparison test, $\int_2^\infty \frac{dx}{x-\sqrt{x}}$ diverges.

(2) Use the comparison test to determine whether the following converge:

(a) $\int_0^\infty \frac{dx}{x^4+x}$.

Solution: This integral is improper both because the function is discontinuous at 0 and because the interval is infinitely long. We split it up the interval into $[0,1]$ and $[1,\infty)$, and split the integral accordingly. Note that for $0 < x \leq 1$, $x^4 \leq x$. Adding x to both sides tells us that $x^4 + x \leq 2x$, so that $\frac{1}{x^4+x} \geq \frac{1}{2x}$. We have that:

$$\begin{aligned}\int_0^1 \frac{dx}{2x} &= \lim_{t\to 0}\int_t^1 \frac{dx}{2x} \\ &= \lim_{t\to 0}\left[\frac{\ln x}{2}\right]_t^1 = \infty\end{aligned}$$

Thus the integral diverges (we don't need to look at the other half). If we were to check the other half, we would find that it converges, so in fact this integral is ∞.

(b) $\int_1^\infty \frac{dx}{\sqrt{x}-1}$

Solution: Again the integral is improper in two ways. We only need to look at one of them. Note that $\sqrt{x}-1 \leq \sqrt{x}$ on the interval $[2,\infty)$, so $1/(\sqrt{x}-1) \geq 1/\sqrt{x}$ on this interval. We can check that the integral $\int_2^\infty \frac{dx}{\sqrt{x}}$ diverges, so the integral $\int_2^\infty \frac{dx}{\sqrt{x}-1}$ diverges. Thus the integral $\int_1^\infty \frac{dx}{\sqrt{x}-1}$ diverges. (Again, we don't need to check the other half.)

(c) $\int_1^\infty \frac{dx}{\sqrt{x}(1+x)}$

Solution: Note that $\sqrt{x}(1+x) = \sqrt{x} + x^{3/2}$. We have that if $x \geq 1$, then $x^{3/2} \leq x^{3/2} + \sqrt{x}$ (since $0 \leq \sqrt{x}$). Thus $1/(x^{3/2}) \geq 1/(x^{3/2}+\sqrt{x})$, so

$$\begin{aligned}0 \leq \int_1^\infty \frac{dx}{x^{3/2}+\sqrt{x}} &\leq \int_1^\infty \frac{dx}{x^{3/2}} \\ &= \lim_{t\to\infty}\int_1^\infty \frac{dx}{x^{3/2}} \\ &= \lim_{t\to\infty}\left[2x^{-1/2}\right]_1^t \\ &= 2-0 = 2\end{aligned}$$

Thus the integral converges to some value between 0 and 2.

(3) Show that $\int_{-\infty}^\infty e^{-x^2}dx$ converges. Be careful, since $e^{-x} < e^{-x^2}$ for some x.

Solution 1: Since e^{-x^2} is an even function, $\int_{-\infty}^0 e^{-x^2}dx = \int_0^\infty e^{-x^2}dx$, so it suffices to show that $\int_0^\infty e^{-x^2}dx$ converges. We can break this up into two integrals:

$$\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}$$

The first integral converges because e^{-x^2} is bounded an continuous on $[0,1]$. The second integral converges, by example 2 above.

Solution 2: Break this up into three integrals as follows:

$$\int_{-\infty}^\infty e^{-x^2}dx = \int_\infty^{-1} e^{-x^2}dx + \int_{-1}^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$$

The second integral converges because e^{-x^2} is bounded and continuous on $[-1,1]$. The third integral converges by example 2 above. For the first integral note that if $x \leq -1$, then $x^2 \geq -x$. Applying the decreasing function e^-y to both sides, we see that $0 \leq e^{-x^2} \leq e^x$. Thus by the comparison test,

$$\begin{aligned} 0 \leq \int_{-\infty}^{-1} e^{-x^2} dx &\leq \int_{-\infty}^{-1} e^x dx \\ &= \lim_{t \to -\infty} [e^x]_t^{-1} \\ &= e^{-1} - 0 = 1/e \end{aligned}$$