WORKSHEET FOR 3/6/2009

Reading assignment for Monday. Read section 11.1. If you will take statistics at some point, reading 10.3 might be a good idea.
Homework due Monday. 10.2: 8, 10, 18, 19, 20, 22
Notes: The comparison test lets you test whether a given integral converges without actually computing it. It can also be used to estimate the values of integrals by using the values of integrals we can compute.

Theorem. *Let f and g be continuous on (a, b) and that $0 \leq f(x) \leq g(x)$ for all x in (a, b). Then*

- *If $\int_a^b g(x)dx$ converges, then so does $\int_a^b f(x)$, and also:*

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

- *If $\int_a^b f(x)dx$ diverges, then so does $\int_a^b g(x)dx$.*

Also useful for dealing with functions that take negative values:

Theorem. *If $\int_a^b |f(x)|dx$ converges, then $\int_a^b f(x)dx$ converges and $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$.*

When dealing with inequalities, it's useful to remember some basics:

- (Triangle inequality) $|f(x) + g(x)| \leq |f(x)| + |g(x)|$
- If $0 < c \leq d$, then $1/c \geq 1/d$.
- If f is an increasing function and $x \leq y$, then $f(x) \leq f(y)$. If f is a decreasing function and $x \leq y$, then $f(x) \geq f(y)$.

Example 1: Determine whether the following integral converges:

$$\int_1^\infty \frac{1}{x^2 - 1} dx$$

Note that $x^2 - 1 \geq x^2/2$ for $x \geq 2$. Thus $\frac{1}{x^2-1} \leq \frac{1}{2x^2}$, and so:

$$\int_2^\infty \frac{dx}{x^2 - 1} \leq \int_2^\infty \frac{dx}{2x^2} = \frac{1}{2} \int_2^\infty \frac{dx}{x^2}$$

We can show this last integral converges using the methods we learned last time.
Example 2: Determine whether the following integral converges: $\int_1^\infty e^{-x^2} dx$.
First note that $x \leq x^2$ when $x \geq 1$, and e^{-y} is a decreasing function, so $e^{-x^2} \leq e^{-x}$ for all $x \geq 1$. Thus by the comparison test

$$\int_1^\infty e^{-x^2} dx \leq \int_1^\infty e^{-x} dx = \lim_{t \to \infty} \left[-e^{-x}\right]_1^t = 1/e$$

Exercises:

(1) Does $\int_2^\infty \frac{dx}{x - \sqrt{x}}$ converge? Hint: $x - \sqrt{x} \leq x$.
(2) Use the comparison test to determine whether the following converge:
  (a) $\int_0^\infty \frac{dx}{x^4 + x}$.
  (b) $\int_1^\infty \frac{dx}{\sqrt{x} - 1}$
  (c) $\int_1^\infty \frac{dx}{\sqrt{x}(1 + x)}$
(3) Show that $\int_{-\infty}^\infty e^{-x^2} dx$ converges. Be careful, since $e^{-x} < e^{-x^2}$ for some x.