WORKSHEET FOR 3/13/2009

Reading assignment for Monday. Review section 11.3 (ratio test)
Homework due Monday. 11.3: 2, 3, 4, 10, 18, 44
Today we're studying ways of bounding series to show convergence.

Theorem. *(Comparison test) Suppose that $0 \le a_k \le b_k$ for all k. If the series $\sum_{k=1}^{\infty} b_k$ converges, then so does $\sum_{k=1}^{\infty} a_k \le \sum_{k=1}^{\infty} b_k$. If $\sum_{k=1}^{\infty} a_k$ diverges, then so does $\sum_{k=1}^{\infty} b_k$.*

Example: Consider the sequence $a_k = k^{-1/2}$. Then $0 \le k^{-1} \le a_k$. We saw last time that the sequence $\sum_{k=1}^{\infty} k^{-1}$ diverges, so the sequence $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges.

Example: Consider the sequence $a_k = 2^{-k^2+1}$. Note that $0 \le 2^{-k^2+1} \le 2^{-k}$ for $k \ge 2$. Thus $\sum_{k=2}^{\infty} 2^{-k^2+1} \le \sum_{k=2}^{\infty} 2^{-k}$, which converges since it is a geometric series, and $\frac{1}{2} < 1$. Thus $\sum 2^{-k^2+1}$ converges.
We can also use integrals to check whether certain series converge, as we saw last time.

Theorem. *(Integral Test) Let $a(x)$ be a continuous, decreasing positive function, and $a_k = a(k)$. Then $\int_1^{\infty} a(x)dx$ converges if and only if $\sum_{k=1}^{\infty} a_k$ converges. If they converge, then*

$$\int_1^{\infty} a(x)dx \le \sum_{k=1}^{\infty} a_k \le a_1 + \int_1^{\infty} a(x)dx \quad \textit{and also:} \sum_{k=n+1}^{\infty} a_k \le \int_n^{\infty} a(x)dx$$

Example: We can easily check that $\int_1^{\infty} \frac{dx}{2x+1}$ diverges. Thus $\sum_{k=1}^{\infty} \frac{1}{2k+1}$ diverges.

Exercises:

(1) Use the integral test to show that $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $0 < p \le 1$.

(2) Use comparison test to show that $\sum_{k=3}^{\infty} \frac{1}{(\ln k)^k}$ converges. (Hint: geometric series.)

(3) Use the integral test to find upper and lower bounds for $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$.

(4) Consider the series $\sum_{k=1}^{\infty} \frac{e^{\sin k}}{k^2}$.
  (a) $\int_1^{\infty} (e^{\sin x})/x^2 dx$ is a convergent improper integral. Explain why it does not follow that the series converges. (In other words, explain why the integral test does not apply. Justify your answer.)
  (b) Show that the series converges. (Hint: $\sin k \le 1$.)