

WORKSHEET FOR 4/3/2009

Reading assignment. Section 11.6

Homework due Wednesday. 7, 9, 12, 14, 21, 22, 26, 27, 28, 39, 40 (Wednesday is not a typo)

Notes: Power series are series of the form:

$$S(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k$$

Here x is a variable and x_0 is a constant, and it's important to note that it does not depend on the index of the sum k . These series are very useful in that they allow us to express many functions as infinite polynomials, which (as we'll see on Monday) are easy to differentiate and integrate. This makes some properties of functions easy to understand that were hard to see before.

One important thing to know about a power series is the interval on which it converges. Our most powerful tool here is the ratio test, though the comparison test and the alternating series test can come in handy too.

Theorem. *Let $S(x)$ be as above. The set of values for which $S(x)$ converges is an interval centered at x_0 . In other words, S converges only on an interval of the form $(x_0 - r, x_0 + r)$, $(x_0 - r, x_0 + r]$, $[x_0 - r, x_0 + r)$ or $[x_0 - r, x_0 + r]$.*

We can differentiate or integrate power series just as we do for polynomials, but we need to take note of where the new series converges.

Example: Consider $\sum_{k=0}^{\infty} x^k$. We already know that this series converges if $|x| < 1$ and diverges otherwise, and in fact we know its limit: $1/(1 - x)$. Calculating its derivative, we get:

$$\frac{d}{dx} \sum_{k=0}^{\infty} x^k = \sum_{k=1}^{\infty} kx^{k-1}$$

We can figure out where this series converges by looking at the ratio test:

$$\lim_{k \rightarrow \infty} \frac{(k+1)x^k}{kx^{k-1}} = \lim_{k \rightarrow \infty} \frac{(k+1)x}{k} = x$$

Thus the derivative converges exactly on the interval $(-1, 1)$, just as we would hope. This gives us the following equation:

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}$$

Exercises:

(1) Find the radii of convergence of the following series, and find their value on the interval:

(a) $\sum_{k=1}^{\infty} 5(x-5)^k$

(b) $\sum_{k=1}^{\infty} k! \cdot x^k$

(2) Find the radius of convergence for the following series:

(a) $1 + (x-2)^2 - 5(x-2)^{10}$

(b) $\sum_{n=2}^{\infty} \frac{(x-3)^{2n}}{n^4}$

(3) Find the derivative and antiderivatives of the following series. Determine where the series converges and where the antiderivative converges.

(a) $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

(b) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$.

(c) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$