Math 415 Exam I

Calculators, books and notes are not allowed!

Name: ______________________________

Student ID: ______________________________

Math 415 Exam I

Score: ______________________________

(20pts) 1. Let A be a square matrix satisfying $A^2 = 2A$. Find the determinant of A.

Sol. From $A^2 = 2A$, we get

$$(\det A)(\det A) = \det(2A) = 2^n \det A\,,$$

where n is the number of columns in A. Thus $\det A = 0$ or $\det A = 2^n$.

(20pts) 2. Let $\mathcal{M}_{3\times 3}$ be the vector space of all real matrices of size 3×3, equipped with the matrix addition and scalar multiplication. Let W be the set of all 3×3 upper triangular real matrices. Prove that W is a subspace of $\mathcal{M}_{3\times 3}$.

Proof. We only need to prove that W is closed under the matrix addition and scalar multiplication.

First, let $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$, $\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix}$ be any two matrices in W. From the definition of the matrix addition,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ 0 & a_{22}+b_{22} & a_{23}+b_{23} \\ 0 & 0 & a_{33}+b_{33} \end{pmatrix},$$

which is an upper triangular matrix in W.

Second, let c be any real number (a scalar). From the definition of the scalar multiplication,

$$c \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ 0 & ca_{22} & ca_{23} \\ 0 & 0 & ca_{33} \end{pmatrix},$$

which is an upper triangular matrix in W.

Therefore W is closed under the matrix addition and scalar multiplication and we prove that W is a subspace of $\mathcal{M}_{3\times 3}$.

□

(20pts) 3. Find the LU factorization of $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & 3 \\ 0 & 4 & 4 \end{pmatrix}$.

Sol. Perform the following elementary row operations for A:
1) Add $(-2)row\,1$ to $row\,2$,
2) Add $(-1)row\,2$ to $row\,3$.

By operations 1) and 2), A can be reduced to $U = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}$. Let L_1 be the elementary matrix associated to operation 1). And Let L_2 be the elementary matrix associated to operation 2). Then

$$L_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } L_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Thus

$$A = L_1^{-1}L_2^{-1}U = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix},$$

which yields the LU factorization of A.

(25pts)4. Let $A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & 2 & 3 & -1 \\ 0 & 4 & 3 & -3 \end{pmatrix}$.

a) Find a basis for Range$A$.
b) Find a basis for Ker$A$.
c) Find the general solution to the linear system $A\vec{x} = \begin{pmatrix} 0 \\ b \\ b \end{pmatrix}$, where b is a real number.

Sol. a) Perform the following elementary row operations for A:
1) Add $(-2)row\,1$ to $row\,2$,
2) Add $(-1)row\,2$ to $row\,3$.

By operations 1) and 2), A can be reduced to $U = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 4 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Since the pivots 1 and 4 are in the first column and the second column respectively, the first column and the second column in A are linearly independent. Therefore a basis of Range$A$ is $\{(1,2,0)^T, (-1,2,4)^T\}$.

b) To obtain Ker$A$, we need to solve the homogeneous linear system $U\vec{x} = \vec{0}$. This homogeneous liner system can be written as

$$\begin{cases} x_1 - x_2 + x_4 = 0 \\ 4x_2 + 3x_3 - 3x_4 = 0 \end{cases}$$

Then we get $x_1 = -\frac{3}{4}x_3 - \frac{1}{4}x_4$ and $x_2 = -\frac{3}{4}x_3 + \frac{3}{4}x_4$ for fixed x_3, x_4. Thus

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{3}{4}x_3 - \frac{1}{4}x_4 \\ -\frac{3}{4}x_3 + \frac{3}{4}x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -3/4 \\ -3/4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1/4 \\ 3/4 \\ 0 \\ 1 \end{pmatrix}$$

The fundamental theorem of Linear Algebra yields dim(Ker$A$) $= 4 -$ dim(Range$A$) $= 4 - 2 = 2$. Thus a basis of Ker(A) is $\{(-3/4, -3/4, 1, 0)^T, (1/4, 3/4, 0, 1)^T\}$.

c) Since we know Ker$A$ is span$\{(-3/4, -3/4, 1, 0)^T, (1/4, 3/4, 0, 1)^T\}$ from part b), we only need to find a particular solution to $A\vec{x} = (0, b, b)^T$ in order to obtain the general solution. By performing the operations 1) and 2) in part a), we reduce the augmented matrix $\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 2 & 2 & 3 & -1 & b \\ 0 & 4 & 3 & -3 & b \end{array}\right)$ to $\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 4 & 3 & -3 & b \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$. The corresponding linear system is

$$\begin{cases} x_1 - x_2 + x_4 = 0 \\ 4x_2 + 3x_3 - 3x_4 = b \end{cases}$$

Clearly $\vec{x} = (b/4, b/4, 0, 0)^T$ is a particular solution to this linear system. Thus the general solution is

$$\vec{x} = (x_1, x_2, x_3, x_4)^T = (b/4, b/4, 0, 0)^T + c_1(-3/4, -3/4, 1, 0)^T + c_2(1/4, 3/4, 0, 1)^T.$$

(15pts) 5. Let A be an invertible matrix of $n \times n$. And let $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}$ be (column) vectors in $\mathbb{R}^n$.
a) Prove that if $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}$ are linearly independent, then $A\vec{v_1}, A\vec{v_2}, \cdots, A\vec{v_k}$ are linearly independent.
b) Let $M_1 = \big(\vec{v_1} \cdots \vec{v_k}\big)$ (the matrix generated by $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}$), and $M_2 = \big(A\vec{v_1} \cdots A\vec{v_k}\big)$ (the matrix generated by $A\vec{v_1}, A\vec{v_2}, \cdots, A\vec{v_k}$). Prove that $\text{rank}M_1 = \text{rank}M_2$.

Proof. Proof of Part a).
Let $c_1 A\vec{v_1} + c_2 A\vec{v_2} + \cdots c_k A\vec{v_k} = \vec{0}$, where $c_1, c_2, \cdots, c_k$ are scalars. To prove the linear independence of $A\vec{v_1}, A\vec{v_2}, \cdots, A\vec{v_k}$, we only need to prove $c_1 = c_2 = \cdots = c_k = 0$. Notice that

$$c_1 A\vec{v_1} + c_2 A\vec{v_2} + \cdots c_k A\vec{v_k} = A(c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k})\,.$$

Thus we obtain

$$A(c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k}) = \vec{0}\,. \tag{1}$$

Since A is invertible, A^{-1} exists. Multiply both sides of (1) by A^{-1} to get

$$A^{-1}A(c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k}) = A^{-1}\vec{0}\,,$$

which yields

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} = \vec{0}\,.$$

Since in part a) $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}$ are linearly independent, we get that $c_1 = c_2 = \cdots = c_k = 0$ from the definition of linear independence. Thus the proof of part a) is completed.

Proof of Part b).

Method 1. From Part a), one can conclude that $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}$ are linearly independent if and only if $A\vec{v_1}, A\vec{v_2}, \cdots, A\vec{v_k}$ are linearly independent. This is because $\vec{v_j} = A^{-1}(A\vec{v_j})$ for all j. Thus we can conclude $\text{rank}M_1 = \text{rank}M_2$ from the definition of the rank.

Method 2. Set $r_1 = \text{rank}M_1$ and $r_2 = \text{rank}M_2$. First, we prove $r_1 \leq r_2$. From the definition of the rank, we know there are r_1 linearly independent vectors among $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}$. Let us name these r_1 linearly independent vectors by $\vec{u_1}, \cdots, \vec{u_{r_1}}$ respectively. From Part a), $A\vec{u_1}, \cdots, A\vec{u_{r_1}}$ are linearly independent. Thus we get r_1 linearly independent vectors among $A\vec{v_1}$, $A\vec{v_2}$, $\cdots$, $A\vec{v_k}$. Thus the maximal number of linearly independent vectors among $A\vec{v_1}$, $A\vec{v_2}$, $\cdots$, $A\vec{v_k}$ must at least be r_1, which gives $r_1 \leq r_2$ by the definition of the rank.

Now we prove $r_2 \leq r_1$. This can be done similarly. In fact, let $\vec{w_1} = \overrightarrow{Av}_1$, $\vec{w_2} = \overrightarrow{Av}_2$, $\cdots$, $\vec{w_k} = Av_k$. Since A is invertible, we have $v_1 = A^{-1}w_1$, $v_2 = A^{-1}w_2$,

$\cdots$, $v_k = A^{-1}w_k$. Repeat the same argument as in the previous paragraph to conclude $\text{rank}(\vec{w_1} \cdots \vec{w_k}) \leq \text{rank}(A^{-1}\vec{w_1} \cdots A^{-1}\vec{w_k})$, which gives $r_2 \leq r_1$ since $(\vec{w_1} \cdots \vec{w_k}) = M_2$ and $(A^{-1}\vec{w_1} \cdots A^{-1}\vec{w_k}) = M_1$.

Finally we conclude $r_1 = r_2$ since $r_1 \leq r_2$ and $r_2 \leq r_1$. And this completes the proof. $\square$