Math 415 Exam II

Calculators, books and notes are not allowed!

Name: ______________________________

Student ID: ______________________________

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Score: ______________________________

(25pts) 1. Find the minimizer and minimum value for the following function.

$$p(x, y, z) = x^2 + 6y^2 + 9z^2 + 4xy - 2xz - 4x - 16y - 16z + 10\,.$$

Sol. Let $K = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 6 & 0 \\ -1 & 0 & 9 \end{pmatrix}$ and $\vec{f} = \begin{pmatrix} 2 \\ 8 \\ 8 \end{pmatrix}$.

First, verify that K is a positive definite matrix. This can be done by performing elementary row operations (of type 1) to reduce K to $U = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 6 \end{pmatrix}$. Since the diagonal entries in U are all positive, K is positive definite.

We now solve the linear system $K\vec{x} = \vec{f}$. Perform elementary row operations for the augmented matrix $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 6 & 0 & 8 \\ -1 & 0 & 9 & 8 \end{array}\right)$ to obtain the upper triangular matrix $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 6 & 6 \end{array}\right)$. Then employ the back-substitution method to solve the corresponding linear system

$$\begin{cases} x_1 + 2x_2 - x_3 & = & 2 \\ 2x_2 + 2x_3 & = & 4 \\ 6x_3 & = & 6 \end{cases}$$

Thus we get $x_1 = x_2 = x_3 = 1$, which yields the minimizer $(1, 1, 1)^T$. Finally, the minimum value for the function is $p(1, 1, 1) = -8$.

(25pts) 2. Find the point on the plane $x + y - z = 0$ that is closest to $\vec{b} = (1, 1, 1)^T$.

Sol. First we find a basis for the plane.
$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis since the two vectors in the set are linearly independent and the dimension of the plane is 2. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, which is the matrix generated by the vectors in the basis. Now set

$$K = A^T A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

and

$$\vec{f} = A^T \vec{b} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Solve the linear system $K \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{f}$ to obtain

$$x_1 = 2/3\,, \text{ and } x_2 = 2/3\,.$$

The closest point $\vec{v}^*$ is

$$\vec{v}^* = x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \\ 4/3 \end{pmatrix}.$$

(30pts)3. Let $\vec{v} = (1, 0, 0, 1)^T$, $A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & -1 & 2 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix}$ and $W = \text{Ker}A$.

a) Find an orthogonal basis for $W^{\perp}$.
b) Find the orthogonal projection of $\vec{v}$ onto W.
c) Represent $\vec{v}$ as $\vec{w} + \vec{z}$, where $\vec{w} \in W$ and $\vec{z} \in W^{\perp}$.

Sol. a) From Section 5.6, we know that $(\text{Ker}A)^{\perp} = \text{corange}A = \text{range}A^T$. Perform the following elementary row operations for $A^T = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & -1 \end{pmatrix}$: (i) add row 1 to row 2; (ii) add (-1)row 1 to row 4; (iii) add row 2 to row 4; (iv) add (-2)row 2 to row 3. Then we reduce A^T to $U = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 2 \\ 1 \end{pmatrix} \right\}$ is a basis for $W^{\perp} = \text{range}A^T$. Let $\vec{w}_1 = (1, -1, 0, 1)^T$ and $\vec{w}_1 = (2, -1, 2, 1)^T$. We now use the Gram-Schmidt process to obtain an orthogonal basis for $W^{\perp}$. Let

$$\vec{v}_1 = \vec{w}_1 = (1, -1, 0, 1)^T ,$$

and

$$\vec{v}_2 = \vec{w}_2 - \frac{\vec{w}_2 \cdot \vec{v}_1}{\|v_1\|^2}\vec{v}_1 = (2, -1, 2, 1)^T - \frac{4}{3}(1, -1, 0, 1)^T = (\frac{2}{3}, \frac{1}{3}, 2, -\frac{1}{3})^T .$$

Thus $\{\vec{v}_1, \vec{v}_2\} = \{(1, -1, 0, 1)^T, (\frac{2}{3}, \frac{1}{3}, 2, -\frac{1}{3})^T\}$ is a basis of $W^{\perp}$.

b) Normalize vectors $\vec{v}_1, \vec{v}_2$ to obtain an orthonormal basis for $W^{\perp}$:

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}})^T , \quad \vec{u}_1 = \frac{\vec{v}_2}{\|\vec{v}_1\|} = (\frac{2}{\sqrt{42}}, \frac{1}{\sqrt{42}}, \frac{6}{\sqrt{42}}, -\frac{1}{\sqrt{42}})^T .$$

The orthogonal projection of $\vec{v} = (1, 0, 0, 1)^T$ onto $W^{\perp}$ is

$$\begin{aligned} \vec{z} &= (\vec{v} \cdot \vec{u}_1)\, \vec{u}_1 + (\vec{v} \cdot \vec{u}_2)\, \vec{u}_2 = \frac{2}{\sqrt{3}}\vec{u}_1 + \frac{1}{\sqrt{42}}\vec{u}_2 \\ &= (\frac{2}{3}, -\frac{2}{3}, 0, \frac{2}{3})^T + (\frac{1}{21}, \frac{1}{42}, \frac{1}{7}, -\frac{1}{42})^T = (\frac{5}{7}, -\frac{9}{14}, \frac{1}{7}, \frac{9}{14})^T . \end{aligned}$$

The orthogonal projection of $\vec{v} = (1, 0, 0, 1)^T$ onto W is

$$\vec{w} = \vec{v} - \vec{z} = (1, 0, 0, 1)^T - (\frac{5}{7}, -\frac{9}{14}, \frac{1}{7}, \frac{9}{14})^T = (\frac{2}{7}, \frac{9}{14}, -\frac{1}{7}, \frac{5}{14})^T .$$

c) Finally we represent $\vec{v} = (1, 0, 0, 1)^T$ as

$$\vec{v} = \vec{w} + \vec{z} = (\frac{2}{7}, \frac{9}{14}, -\frac{1}{7}, \frac{5}{14})^T + (\frac{5}{7}, -\frac{9}{14}, \frac{1}{7}, \frac{9}{14})^T .$$

(20pts) 4. Prove that the following identities hold.

a) For all vectors $\vec{v}$ and $\vec{w}$ in an inner product space,

$$\|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2 = 2\big(\|\vec{v}\|^2 + \|\vec{w}\|^2\big)\,.$$

b) If $\vec{v}$, $\vec{w}$ are orthogonal, then

$$\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2\,.$$

Proof. a) Use the definition of the norm and the bilinearity of the inner product to obtain

$$\|\vec{v}+\vec{w}\|^2 = \langle \vec{v}+\vec{w}, \vec{v}+\vec{w}\rangle = \langle \vec{v},\vec{v}\rangle + \langle \vec{v},\vec{w}\rangle + \langle \vec{w},\vec{v}\rangle + \langle \vec{w},\vec{w}\rangle = \|\vec{v}\|^2 + 2\langle \vec{v},\vec{w}\rangle + \|\vec{w}\|^2\,,$$

and

$$\|\vec{v}-\vec{w}\|^2 = \langle \vec{v}-\vec{w}, \vec{v}-\vec{w}\rangle = \langle \vec{v},\vec{v}\rangle - \langle \vec{v},\vec{w}\rangle - \langle \vec{w},\vec{v}\rangle + \langle \vec{w},\vec{w}\rangle = \|\vec{v}\|^2 - 2\langle \vec{v},\vec{w}\rangle + \|\vec{w}\|^2\,,$$

Thus

$$\begin{aligned} & \|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2 \\ = \; & \|\vec{v}\|^2 + 2\langle \vec{v},\vec{w}\rangle + \|\vec{w}\|^2 + \|\vec{v}\|^2 - 2\langle \vec{v},\vec{w}\rangle + \|\vec{w}\|^2 \\ = \; & 2\|\vec{v}\|^2 + 2\|\vec{w}\|^2 \end{aligned}$$

b) If $\vec{v}$, $\vec{w}$ are orthogonal, then $\langle \vec{v},\vec{w}\rangle = 0$. Thus

$$\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + 2\langle \vec{v},\vec{w}\rangle + \|\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2\,.$$

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