Math 553 Exam II

1. Let $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{H} = (H_1, H_2, H_3)$, where $E_1, E_2, E_3, H_1, H_2, H_3$ are C^2 functions of $(\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}$. Suppose that $\mathbf{E}, \mathbf{H}$ satisfy Maxwell equation

$$\begin{cases} \mathbf{E}_t &= \operatorname{curl} \mathbf{H} \\ \mathbf{H}_t &= -\operatorname{curl} \mathbf{E} \\ \operatorname{div} \mathbf{E} &= \operatorname{div} \mathbf{H} = 0 \,. \end{cases}$$

Show that $E_1, E_2, E_3, H_1, H_2, H_3$ satisfy the 3-d wave equation $u_{tt} - \Delta u = 0$.

2. Let $u \in C^2(\mathbb{R} \times [0, \infty))$ solves the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = g(x), \;\; u_t(x,0) = h(x) \,, \end{cases}$$

where g, h are supported on the interval $[-R, R]$. The kinetic energy is

$$K(t) = \frac{1}{2} \int_{\mathbb{R}} u_t^2(x,t) dx$$

and the potential energy is

$$P(t) = \frac{1}{2} \int_{\mathbb{R}} u_x^2(x,t) dx \,.$$

Prove that
a) $K(t) + P(t)$ is constant in t.
b) (Equipartition of energy) $K(t) = P(t)$ for all large enough times t.

3. Consider the n-dimensional wave equation with dissipation

$$\begin{cases} u_{tt} - \Delta u + \alpha u_t = 0 \\ u(x,0) = g(x), \;\; u_t(x,0) = h(x) \,, \end{cases} \tag{1}$$

where g and h are supported on the ball $B(0, R)$ and $\alpha \geq 0$ is a constant. Show that if u is a solution of (1), then for fixed t, $u(\cdot, t)$ is a function with a compact support (whose size depends on R).

Hint: *Prove any solution of (1) has a finite propagation speed, that is, for any* $x_0 \in \mathbb{R}^n$ *and* $t_0 > 0$, *if* $f \equiv g \equiv 0$ *in* $B_0 = \{x \in \mathbb{R}^n : |x - x_0| \leq t_0\}$, *then* $u(x_0, t_0) = 0$. *To prove it, first define a local energy by*

$$E(t) = \frac{1}{2} \int_{B(x_0, t_0 - t)} (u_t^2 + |\nabla u|^2) dx \,.$$

And then prove E *is decreasing.*