Math 553 Final Exam

1. Let $x \in \mathbb{R}$, $t > 0$ and $u(x,t) = v\left(x^2/t\right)$.
a) Show that $u_t = u_{xx}$ if and only if

$$4zv''(z) + (2+z)v'(z) = 0, \quad z > 0\,. \tag{1}$$

b) Show that the general solution of (1) is

$$v(z) = c_1 \int_0^z e^{-s/4} s^{-1/2} ds + c_2\,,$$

where c_1, c_2 are constants.
c) Differentiate $v\left(x^2/t\right)$ with respect to x and select the constant c_1 properly, so as to obtain the 1-d heat kernel.

2. Let $n \geq 3$. Show that $K(x) = \frac{1}{(2-n)\omega_n |x|^{n-2}}$ is the fundamental solution for the Laplace operator.

3. a) Use Fourier transform to derive a formula for the solution of Schrödinger's equation

$$\begin{cases} iu_t + \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty); \\ u(x,0) = g(x) & \text{for all } \; x \in \mathbb{R}^n\,. \end{cases} \tag{2}$$

Here u and g are complex-valued.
b) Use Part a) to show that if u is a solution of the Schrödinger equation (2), then

$$\|u(\cdot,t)\|_\infty \leq \frac{1}{(4\pi|t|)^{n/2}} \|g\|_1\,,$$

for each $t > 0$.