

Motivic symmetric ring spectrum representing algebraic K -theory

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Introduction

- Morel and Voevodsky – \mathbb{A}^1 -homotopy theory of schemes
- Voevodsky's spectrum BGL represents algebraic K -theory:

$$BGL^{p,q}(X_+) = K_{2q-p}(X)$$

for any smooth scheme X over a regular base scheme S .

- Jardine's category of motivic symmetric spectra

$$\begin{array}{ccc} SM^\Sigma(S) & \xrightarrow[\text{Quillen equivalence}]{\text{forgetful functor}} & SM(S) \\ \downarrow & & \downarrow \\ SH^\Sigma(S) & \xrightarrow{\sim} & SH(S) \end{array}$$

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- Röndigs, Spitzweck, Østvær – Motivic strict ring models for K -theory (K-theory preprint archive, 2009)

Definition

A simplicial presheaf on $(Sm/S)_{Nis}$ is called a *motivic space*.

Definition (Jardine)

A *motivic symmetric T -spectrum over S* is

- 1 a sequence $A_0, A_1, \dots, A_n, \dots$ of pointed motivic spaces.
- 2 a pointed map $\sigma : T \wedge A_n \rightarrow A_{1+n}$ for each $n \geq 0$, and
- 3 a (base point preserving) left action of the symmetric group Σ_n on A_n for $n \geq 0$, such that the composite map

$$\sigma^p : T^p \wedge A_n \rightarrow T^{p-1} \wedge A_{1+n} \rightarrow \dots \rightarrow T \wedge A_{p-1+n} \rightarrow A_{p+n}$$

of the maps $Id_{T^i} \wedge \sigma : T^i \wedge T \wedge A_{p-i-1+n} \rightarrow T^i \wedge A_{p-i+n}$ is $\Sigma_p \times \Sigma_n$ -equivariant for $p \geq 0$ and $n \geq 0$.

Standard Vector Bundles

Definition

A free R -module of finite rank of the form

$$R^n = \{(r_1, \dots, r_n) \mid r_i \in R \text{ for } i = 1, \dots, n\}$$

is called a *standard free R -module*.

Definition

Let X be a scheme. A *standard vector bundle on X* is a pair (H, E) where H is a sieve on X that belongs to the big Zariski site on X and E is a presheaf of modules defined only on H such that for any object $f : Y \rightarrow X$ of H , $E(Y)$ is a standard free $\mathcal{O}_X(Y)$ -module of rank n for some integer n , and the map $\mathcal{O}_Y^n \rightarrow f^*E$ induced by the standard basis of $E(Y)$ is an isomorphism. A *morphism* between two standard vector bundles is a map between their associated sheaves.

Standard Vector Bundles

$\mathbf{V}(X)$: the category of *standard vector bundles* on X

- $\mathbf{V}(X)$ is equivalent to the category of vector bundles on X .
- $\mathbf{V}(X)$ is a small exact category.
- Direct sum $\oplus : \mathbf{V}(X) \times \mathbf{V}(X) \rightarrow \mathbf{V}(X)$
- Tensor product $\otimes : \mathbf{V}(X) \times \mathbf{V}(X) \rightarrow \mathbf{V}(X)$
- Pullback functor $f^* : \mathbf{V}(X) \rightarrow \mathbf{V}(Y)$ for each $f : Y \rightarrow X$

- Strict associativity

$$\begin{array}{ccc}
 \mathbf{V}(X) \times \mathbf{V}(X) \times \mathbf{V}(X) & \xrightarrow{\otimes \times 1} & \mathbf{V}(X) \times \mathbf{V}(X) \\
 \downarrow 1 \times \otimes & & \downarrow \otimes \\
 \mathbf{V}(X) \times \mathbf{V}(X) & \xrightarrow{\otimes} & \mathbf{V}(X)
 \end{array}$$

- Strict commutativity with line bundles

$$\begin{array}{ccc}
 \mathbf{V}(X) \times \mathbf{L}(X) & \xrightarrow{\tau} & \mathbf{L}(X) \times \mathbf{V}(X) \\
 \searrow \otimes & & \swarrow \otimes \\
 & \mathbf{V}(X) &
 \end{array}$$

- Strict functoriality

$$\begin{array}{ccc}
 \mathbf{V}(X) \times \mathbf{V}(X) & \xrightarrow{\otimes} & \mathbf{V}(X) \\
 f^* \times f^* \downarrow & & \downarrow f^* \\
 \mathbf{V}(Y) \times \mathbf{V}(Y) & \xrightarrow{\otimes} & \mathbf{V}(Y)
 \end{array}$$

for $f : Y \rightarrow X$

$$\begin{array}{ccc}
 \mathbf{V}(X) & \xrightarrow{f^*} & \mathbf{V}(Y) \\
 & \searrow (fg)^* & \swarrow g^* \\
 & \mathbf{V}(Z) &
 \end{array}$$

for $Z \xrightarrow{g} Y \xrightarrow{f} X$

- $G\mathbf{V}(X)$: algebraic K -theory space of $\mathbf{V}(X)$

$$|G\mathbf{V}(X)| \simeq \Omega|S.\mathbf{V}(X)|$$

$$\pi_i(G\mathbf{V}(X)) \cong K_i(X) \quad \text{for all } i \geq 0$$

- $G^r\mathbf{V}(X)$: iterated G -construction (multisimplicial set)

$$|G\mathbf{V}(X)| \simeq |G^2\mathbf{V}(X)| \simeq \cdots \simeq |G^r\mathbf{V}(X)| \simeq \cdots$$

- A simplex of $G^r\mathbf{V}(X)$ is represented by filtrations of 2^r objects of $\mathbf{V}(X)$ arranged in an r -dimensional cube such that the subquotients of the filtrations satisfy certain compatibility conditions.

Example

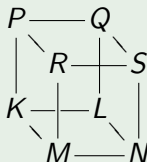
$r = 1$

P, Q

$r = 2$

$P \quad Q$
 $R \quad S$

$r = 3$



- Σ_r acts on $G^r\mathbf{V}(X)$ by permuting the coordinates of the cube.

- Product map

$$\otimes : G^p \mathbf{V}(X) \times G^q \mathbf{V}(X) \rightarrow G^{p+q} \mathbf{V}(X)$$

Example

$$(P, Q) \otimes (M, N) = \begin{pmatrix} P \otimes M & Q \otimes M \\ P \otimes N & Q \otimes N \end{pmatrix}$$

Definition of the motivic spectrum \mathcal{K}

- $T =$ mapping cylinder of the inclusion map $S \rightarrow \mathbb{P}_S^1$ into ∞ .
- $\mathcal{K}_0 = S_+$, $\mathcal{K}_n = X \mapsto \text{diag } G^n \mathbf{V}(X)$
- $\sigma : T \wedge \mathcal{K}_n \rightarrow \mathcal{K}_{1+n}$ is defined by multiplication with $[\mathcal{O}] - [\mathcal{O}(-1)] \in K_0(\mathbb{P}^1)$.

For example, if $f : X \rightarrow \mathbb{P}_S^1 \in T(X)$ and $(P, Q) \in \mathcal{K}_1(X)$, then σ sends them to

$$(\mathcal{O}_X, L_f) \otimes (P, Q) = \begin{pmatrix} P & L \otimes P \\ Q & L \otimes Q \end{pmatrix}$$

where $L_f = f^*(\mathcal{O}_S(-1))$.

Theorem

The motivic spectrum \mathcal{K} is a motivic symmetric ring spectrum.

- symmetric group action
- equivariance of σ^P
- multiplicative structure

$$\begin{aligned}\sigma^2 : T(X) \wedge T(X) &\rightarrow \mathcal{K}_2(X) \\ (f, g) &\mapsto \begin{pmatrix} \mathcal{O}_X & L_f \\ L_g & L_f \otimes L_g \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\tau = (1\ 2) \in \Sigma_2, \quad \sigma^2 \tau(f, g) &= \begin{pmatrix} \mathcal{O}_X & L_g \\ L_f & L_g \otimes L_f \end{pmatrix} \\ \tau \sigma^2(f, g) &= \begin{pmatrix} \mathcal{O}_X & L_g \\ L_f & L_f \otimes L_g \end{pmatrix}\end{aligned}$$

- Voevodsky's BGL is a motivic \mathbb{P}^1 -spectrum.

$$(BGL)_n = Ex^{\mathbb{A}^1}(\mathbb{Z} \times Gr(\infty, \infty))$$

$$\sigma : \mathbb{P}_S^1 \wedge (BGL)_n \xrightarrow{[\mathcal{O}] - [\mathcal{O}(-1)]} (BGL)_{1+n}$$

- $(BGL)_n$ and \mathcal{K}_n are motivically equivalent for $n \geq 1$.

$$\begin{array}{ccc}
 T \wedge \mathcal{K}_n & \xrightarrow{[\mathcal{O}] - [\mathcal{O}(-1)]} & \mathcal{K}_{1+n} \\
 \downarrow \simeq & & \downarrow \simeq \\
 \mathbb{P}^1 \wedge (BGL)_n & \xrightarrow{[\mathcal{O}] - [\mathcal{O}(-1)]} & (BGL)_{1+n}
 \end{array}$$