MATH 385-X1 DIFFERENTIAL EQUATIONS: FINAL EXAM

FALL, 2004

Name : ____________________

(1) [$8 = 2 + 2 + 2 + 2$ points] Solve the following differential equations and sketch the solutions. Primes denote derivatives with respect to x.

(a) $y' = 2y$ with $y(0) = 1$

(b) $y' = 2x$ with $y(0) = 1$

(c) $y'' = 4y$ with $y(0) = 1,\ y'(0) = 0$

(d) $y'' = -4y$ with $y(0) = 1,\ y'(0) = 0$

(2) [14 = 7 + 7 points] Consider the following first order differential equation:

$$x^2y' + 2xy = 3y^2, \ y(1) = \frac{1}{2}. \tag{1}$$

(a) First, you consider this DE as a Bernoulli equation to find the solution.

(b) Second, you consider this DE (1) as a homogeneous equation and find the solution. You need to use partial fractions. You must have the same answer as in (a).

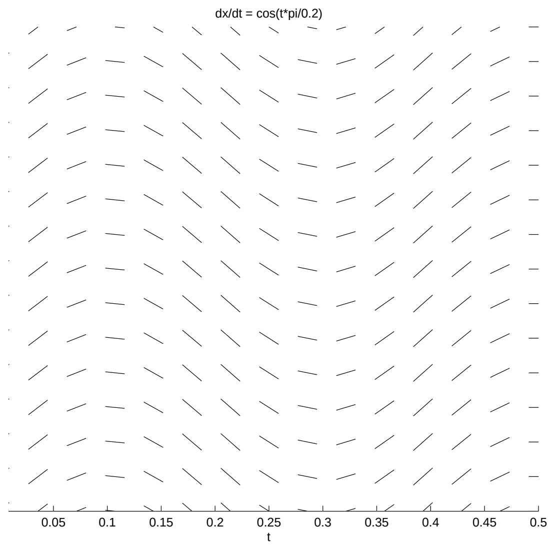
(3) [$10 = 2+2+2+2+2$ points] The Euler update formula for $\dfrac{dx}{dt} = f(t,x)$ says that

$$t_{i+1} = t_i + h \text{ and } x_{i+1} = x_i + hk_1, \text{ where } k_1 = f(t_i, x_i).$$

The Improved Euler update formula says that

$$t_{i+1} = t_i + h \text{ and } x_{i+1} = x_i + h(k_1+k_2)/2, \text{ where } k_1 = f(t_i,x_i) \text{ and } k_2 = f(t_{i+1}, x_i + hk_1).$$

(a) Label each sketch clearly.



On the the direction field above, take $t_0 = 0,\ x_0 = 0.25$ and

(i) sketch the solution curve,

(ii) sketch the first step of the Euler method with $h = 0.1$,

(iii) sketch the first step of the Improved Euler method with $h = 0.1$.

(b) Using the update formulas with $\dfrac{dx}{dt} = t^2 - x + 1$, take $t_0 = 1,\ x_0 = 2$ and

(i) calculate t_1 and x_1 with the Euler method with $h = 1$,

(ii) calculate t_1 and x_1 with the Improved Euler method with $h = 1$.

(4) [$14 = 8 + 4 + 2$ points]

(a) Show by using Undetermined Coefficients that a particular solution of the damped, forced oscillator equation

$$x'' + 2x' + 6x = \cos(\omega t)$$

is

$$x_p(t) = \frac{6-\omega^2}{(6-\omega^2)^2 + (2\omega)^2}\cos(\omega t) + \frac{2\omega}{(6-\omega^2)^2 + (2\omega)^2}\sin(\omega t).$$

(b) The solution in (a) can also be written $x_p(t) = C(\omega)\cos(\omega t - \alpha)$ where

$$C(\omega) = \frac{1}{\sqrt{(6-\omega^2)^2 + (2\omega)^2}}.$$

Show that practical resonance occurs at $\omega = 2$.

(c) Sketch the solution, $x = x_c + x_p$, for large t assuming $\omega = 2$ and $x(0) = 2,\ x'(0) = 0$.

(5) [10 = 3 + 2 + 5 points] Consider the eigenvalue problem

$$\begin{cases} y'' + \lambda y = 0, \\ y(0) = 0, \ y'(1) = 0. \end{cases}$$

Show that

(a) there are no negative eigenvalues,

(b) zero is not an eigenvalue,

(c) there are only positive eigenvalues $\lambda_n = \left(\frac{(2n-1)\pi}{2}\right)^2$ with associated eigenfunctions $y_n(x) = \sin\left(\frac{(2n-1)\pi x}{2}\right)$, $n = 1, 2, \cdots$.

(6) [10 = 1 + 7 + 2 points] Let

$$f(t) = \begin{cases} 0 & \text{if } -1 < t < 0, \\ t & \text{if } 0 < t < 1, \end{cases}$$

and extend f to be 2-periodic.

(a) Sketch the graph of f over two periods.

(b) Show that the Fourier coefficients of f are

$$a_0 = \frac{1}{2}, \ a_n = \frac{(-1)^n - 1}{n^2\pi^2}, \ b_n = -\frac{(-1)^n}{n\pi}.$$

(c) Use the Fourier series of f at $t = 1$ to show that

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

(7) [$10 = 6 + 4$ points] Let $f(t)$ be the 2-periodic function from problem # 6.

(a) Find the general solution, $x = x_c + x_p$, of the periodically forced undamped oscillator equation

$$x''(t) + 4x(t) = f(t).$$

You must determine the coefficients of x_p.
Determine whether resonance will occur or not.

(b) Find the appropriate form of a particular solution, x_p, of

$$x''(t) + 4\pi^2 x(t) = f(t).$$

You do not need to determine coefficients.
Determine whether resonance will occur or not.

(8) [10 = 6 + 4 points] Solve each of the following boundary value problems and find the limit of $u(x,t)$ as $t \to \infty$.

If you remember the solution formulas from class then you can use them; otherwise, you can find the solutions by separation of variables.

(a) $2u_t = u_{xx},\ 0 < x < \pi,\ t > 0$
$u(0,t) = u(\pi,t) = 0$
$u(x,0) = 2$

(b) $u_t = 4u_{xx},\ 0 < x < 1,\ t > 0$
$u_x(0,t) = u_x(1,t) = 0$
$u(x,0) = 3 - 5\cos(4\pi x)$

(9) [$14 = 10 + 4$ points] Consider the wave equation with Dirichlet boundary conditions

$$y_{tt} = a^2 y_{xx}, \quad 0 < x < L$$
$$y(0,t) = y(L,t) = 0, \quad \text{for all } t.$$

(a) Use separation of variables to find a series solution

$$y(x,t) = \sum_{n=1}^{\infty} \left[C_n \cos\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right) + D_n \sin\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \right].$$

If you remember the solution of the eigenvalue problem with Dirichlet boundary conditions, you can use it.

(b) Find the solution with $y(x,0) = 3\sin\frac{2\pi x}{L}$ and $y_t(x,0) = -\sin\frac{4\pi x}{L}$.