MATH 385-X1
DIFFERENTIAL EQUATIONS:
TEST 2 ANSWER KEYS

FALL, 2004

(1) Find the general solution of the following homogeneous linear differential equation with constant coefficients (Hint: You might use that $y = e^x$ is a solution of the DE): [6 pts]

$$y^{(3)} + 4y'' + y' - 6y = 0.$$

Solution

Its characteristic equation is

$$r^3 + 4r^2 + r - 6 = 0.$$

Since $y = e^x$ is a solution of the DE, r=1 is a solution of the characteristic equation. When we divide $r^3 + 4r^2 + r - 6$ by $r - 1$, we get $r^2 + 5r + 6$ as the quotient and 0 as the remainder, and we obtain

$$(r-1)(r^2 + 5r + 6) = 0$$
$$(r-1)(r+2)(r+3) = 0$$
$$r = 1, \ -2, \ -3.$$

Thus the general solution is

$$y = c_1 e^x + c_2 e^{-2x} + c_3 e^{-3x}.$$

(2) Find the appropriate form of a particular solution of the following nonhomogeneous linear differential equation with constant coefficients. You do not need to determine the coefficients: [6 pts]

$$y^{(4)} + y^{(3)} - 2y'' = 3e^{2x} + 4x + 1.$$

Solution

Let us find y_c, first:

$$r^4 + r^3 - 2r^2 = 0$$
$$r^2(r^2 + r - 2) = 0$$
$$r^2(r+2)(r-1) = 0$$
$$r = 0, \ 0, \ -2, \ 1$$
$$y_c = c_1 + c_2 x + c_3 e^{-2x} + c_4 e^x.$$

Since $f(x) = (3e^{2x}) + (4x + 1)$, we want to try $y_p = Ae^{2x} + Bx + C$. But $Bx + C$ duplicates with y_c, we need to multiply x^2 to eliminate duplication. Note that Ae^{2x} does not duplicate with y_c. Thus, the appropriate form of y_p is

$$y_p = Ae^{2x} + x^2(Bx + C).$$

(3) Find a particular solution of the following differential equation: [6 pts]

$$y'' + 9y = \sec 3x.$$

Solution

Since $f(x) = \sec 3x$, we use variation of parameters. To do so, we first find y_c:

$$\begin{aligned} &r^2 + 9 = 0 \\ &r = \pm 3i \\ &y_c = c_1 \cos 3x + c_2 \sin 3x. \end{aligned}$$

Note that given a nonhomogeneous DE, $y'' + p(x)y' + q(x)y = f(x)$,

$$y_p = y_1 \int \frac{-y_2 f}{W(y_1, y_2)}\, dx + y_2 \int \frac{y_1 f}{W(y_1, y_2)}\, dx, \quad W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2,$$

where $y_c = c_1 y_1 + c_2 y_2$.

Since $y_c = c_1 \cos 3x + c_2 \sin 3x$, we let $y_1 = \cos 3x$ and $y_2 = \sin 3x$, then

$$W(y_1, y_2) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3 \sin 3x & 3 \cos 3x \end{vmatrix} = 3\cos^2 3x + 3 \sin^2 3x = 3.$$

Thus,

$$\begin{aligned} y_p &= \cos 3x \int \frac{-\sin 3x \sec 3x}{3} dx + \sin 3x \int \frac{\cos 3x \sec 3x}{3} dx \\ &= -\frac{\cos 3x}{3} \int \tan 3x dx + \frac{\sin 3x}{3} \int dx \\ &= -\frac{\cos 3x}{3} \ln|\sec 3x| \cdot \frac{1}{3} + \frac{\sin 3x}{3} x \\ &= -\frac{\cos 3x}{9} \ln|\sec 3x| + \frac{x \sin 3x}{3} \end{aligned}$$

(4) Suppose that the mass in an unforced mass-spring-dashpot system with $m = 1$, $c = 2$ and $k = 5$ is set in motion with $x(0) = 1$ and $x'(0) = -1 + 2\sqrt{3}$.

(a) Find the position function $x(t)$. [7 pts]

Solution

$$\begin{aligned} &mx'' + cx' + kx = 0 \\ &x'' + 2x' + 5x = 0 \\ &x = -1 \pm \sqrt{1-5} = -1 \pm 2i \\ &x(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t) \\ &x(0) = c_1 = 1 \\ &x'(t) = -e^{-t}(c_1 \cos 2t + c_2 \sin 2t) + e^{-t}(-2c_1 \sin 2t + 2c_2 \cos 2t) \\ &x'(0) = -c_1 + 2c_2 = -1 + 2\sqrt{3} \\ &c_2 = \sqrt{3} \text{ since } c_1 = 1 \\ &x(t) = e^{-t}(\cos 2t + \sqrt{3} \sin 2t) \end{aligned}$$

(b) Find the pseudoperiod. [1 pt]

Solution $\frac{2\pi}{2} = \pi$

(c) Find the equations of the envelope curves. [2 pts]

Solution

When $x(t) = Ce^{-t} \cos(2t - \alpha)$, $C > 0$ and $0 \le \alpha 2\pi$, the equations of the envelope curves are $x = \pm Ce^{-t}$. Here, $C = \sqrt{1^2 + (\sqrt{3})^2} = 2$, and so $x = \pm 2e^{-t}$.

(d) Determine the behavior of $x(t)$ (f.e., decays, grows, oscillates, decays while oscillating or grows while oscillating). [1 pt]

Solution It decays while oscillating.

(e) Find the time lag. [**Extra Credit of 2 pts**]
Solution

$$\begin{aligned} x(t) &= e^{-t}(\cos 2t + \sqrt{3}\sin 2t) \\ &= 2e^{-t}(\frac{1}{2}\cos 2t + \frac{\sqrt{3}}{2}\sin 2t) \\ &= 2e^{-t}\cos(2t - \frac{\pi}{3}) \\ &= 2e^{-t}\cos[2(t - \frac{\pi}{6})], \end{aligned}$$

since $\cos\alpha = \frac{1}{2}$, $\sin\alpha = \frac{\sqrt{3}}{2}$, when $\alpha = \frac{\pi}{3}$. Here, the time lag is $\frac{\pi}{6}$.

(5) Find all non-negative eigenvalues and an associated eigenfunction for each eigenvalue of the following boundary value problem (Determine whether zero is an eigenvalue. If so, write an eigenfunction associated with zero. And then find all positive eigenvalues with eigenfunctions): [8 pts]

$$y'' + \lambda y = 0,\ y'(0) = 0,\ y'(1) = 0.$$

Solution

- $\lambda = 0$:

$$\begin{aligned} &y = c_1 + c_2 x \\ &y' = c_2 \\ &y'(0) = y'(1) = c_2 = 0 \end{aligned}$$

c_1 can be any number, thus there is a non-trivial solution.
$\lambda = 0$ is an eigenvalue with an eigenfunction $y = 1$.

- $\lambda > 0$:

$$\begin{aligned} &y = c_1\cos\alpha x + c_2\sin\alpha x, \quad \alpha = \sqrt{\lambda} > 0 \\ &y' = \alpha(-c_1\sin\alpha x + c_2\cos\alpha x) \\ &y'(0) = \alpha c_2 = 0 \\ &\text{Thus, } c_2 = 0,\ \text{since } \alpha \neq 0. \\ &y'(1) = -\alpha c_1 \sin\alpha = 0 \end{aligned}$$

To get a nontrivial solution, i.e., $c_1 \neq 0$, $\sin\alpha = 0$, i.e., $\alpha = n\pi, n = 1, 2, 3, \cdots$.
Therefore, $\lambda = \alpha^2 = n^2\pi^2$ are eigenvalues with eigenfunctions $y = \cos n\pi x$, $n = 1, 2, 3, \cdots$.

For **Extra Credit of 3 pts**, show that there is no negative eigenvalue for the given boundary value problem.

Solution

If $\lambda < 0$, $y = c_1\cosh\alpha x + c_2\sinh\alpha x$, where $\alpha = \sqrt{-\lambda} > 0$.

$$\begin{aligned} &y' = \alpha(c_1\sinh\alpha x + c_2\cosh\alpha x) \\ &y'(0) = \alpha c_2 = 0,\ \text{thus, } c_2 = 0 \\ &y'(1) = \alpha c_1 \sinh\alpha = 0 \\ &\text{Since } \alpha \neq 0, \sinh\alpha \neq 0 \text{ and so, } c_1 = 0. \end{aligned}$$

Thus $c_1 = c_2 = 0$, i.e., there is only trivial solution. Hence, there is no negative eigenvalue.

(6) Consider the following forced undamped oscillator equation:

$$x'' + 4x = \cos \omega t.$$

(a) Solve the given differential equation (Hint: You must consider two separate cases). [10 pts]

Solution

Note that $x_c = c_1 \cos 2t + c_2 \sin 2t$

- $\omega \neq 2$:

$$x_p = A \cos \omega t \text{ since there is no } x' \text{ term}$$
$$x_p'' = -A\omega^2 \cos \omega t$$
$$(-A\omega^2 + 4A) \cos \omega t = \cos \omega t$$
$$A(4 - \omega^2) = 1, \text{ i.e., } A = \frac{1}{4 - \omega^2}$$
$$x_p = \frac{1}{4 - \omega^2} \cos \omega t$$
$$x = x_c + x_p = x_c = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{4 - \omega^2} \cos \omega t$$

- $\omega = 2$:

$$x_p = t(A \cos 2t + B \sin 2t)$$
$$x_p' = A \cos 2t + B \sin 2t + 2t(-A \sin 2t + B \cos 2t)$$
$$x_p'' = -2A \sin 2t + 2B \cos 2t + 2(-A \sin 2t + B \cos 2t) - 4t(A \cos 2t + B \sin 2t)$$
$$x_p'' + 4x_p = (2B + 2B - 4tA + 4tA) \cos 2t + (-2A - 2A - 4tB + 4tB) \sin 2t = \cos 2t$$
$$4B = 1, \ -4A = 0, \text{ i.e. } A = 0, \ B = \frac{1}{4}$$
$$x_p = \frac{1}{4} t \sin 2t$$
$$x = x_c + x_p = x_c = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{4} t \sin 2t$$

(b) Determine the behavior of the solution $x(t)$ for $\omega = 1$ and $\omega = 2$ (f.e., decays, grows, oscillates, decays while oscillating or grows while oscillating). [2 pts]

Solution

$\omega = 1$: It oscillates.

$\omega = 2$: It grows while oscillating.

(c) Determine the value of ω when the resonance occurs. [1 pt]

Solution

$\omega = 2$, since the amplitude of the solution gets larger and larger without a bound as time increases.